Midterm #3 of ECE301, Prof. Wang's section
6:30–7:30pm Wednesday, April 2, 2014, ME 1061,

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**

2. This is a closed book exam.

3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.

4. Use the back of each page for rough work.

5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

E-mail:

Signature:

Question 1: [30%, Work-out question, Learning Objectives 3, 4, and 5]
Consider a discrete-time signal

$$x[n] = \begin{cases} 2^{-n} & \text{if } -4 \le n \le 3 \\ \text{periodic with period } N = 8 & \end{cases}$$

and let a_k denote its Fourier series coefficients.

1. [8%] Compute the value of a_4.
2. [8%] Compute the value of $\sum_{k=0}^{7}(-1)^k a_k$.
3. [8%] Compute the value of $\sum_{k=0}^{7} |a_k|^2$.

We have another signal $y[n]$, for which the corresponding Fourier series coefficients are

$$b_k = \begin{cases} k & \text{if } 2 \le k \le 4 \\ 0 & \text{if } 0 \le k \le 1 \text{ or } 5 \le k \le 7 \end{cases}$$

Define $z[n] = x[n]y[n]$ and denote the corresponding Fourier series coefficients by c_k.

4. [6%] Write down the expression of c_3 in terms of a_k.

$$a_k = \frac{1}{8}\sum_{n=-4}^{3} 2^{-n} e^{-jk\frac{2\pi}{8}n} = \frac{1}{8}\sum_{n=-4}^{3} 2^{-n} e^{-jk\frac{\pi}{4}n} = \frac{1}{8}\sum_{n=-4}^{3}\left(\frac{1}{2}e^{-jk\frac{\pi}{4}}\right)^n$$

1) $$a_4 = \frac{1}{8}\sum_{n=-4}^{3}\left(\frac{1}{2}e^{-j\pi}\right)^n = \frac{1}{8}\sum_{n=-4}^{3}\left(-\frac{1}{2}\right)^n = \frac{1}{8}\frac{\left(-\frac{1}{2}\right)^{-4} - \left(-\frac{1}{2}\right)^4}{1-\left(-\frac{1}{2}\right)} = \frac{1}{8}\frac{16-\frac{1}{16}}{3/2} =$$

$$= \frac{2 - \frac{1}{128}}{3/2} = \frac{255}{128}\cdot\frac{2}{3} = \frac{85}{64}$$

2) $$\sum_{k=0}^{7}(-1)^k a_k = \sum_{k=0}^{7} e^{-j\pi k} a_k = \sum_{k=0}^{7} a_k e^{-j\frac{2\pi}{8}k\cdot 4} = x[4] = x[-4] = 2^4$$

3) $$\sum_{k=0}^{7}|a_k|^2 = \frac{1}{8}\sum_{n=-4}^{3}|x[n]|^2 = \frac{1}{8}\sum_{n=-4}^{3}|2^{-n}|^2 = \frac{1}{8}\sum_{n=-4}^{3} 2^{-2n} = \frac{1}{8}\sum_{n=-4}^{3}\left(\frac{1}{4}\right)^n =$$

$$= \frac{1}{8}\frac{\left(\frac{1}{4}\right)^{-4} - \left(\frac{1}{4}\right)^4}{1-\frac{1}{4}} = \frac{1}{8}\frac{256 - \frac{1}{256}}{3/4} = \frac{32 - \frac{1}{256\cdot 8}}{3/4} = \frac{256\cdot 8\cdot 32 - 1}{256\cdot 8}\cdot\frac{4}{3} =$$

$$= \frac{256\cdot 8\cdot 32 - 1}{256\cdot 2\cdot 3}$$

$$e^{-j\pi} = \cos(\pi) - j\sin(\pi) = -1$$

4) $c_k = a_k * b_k = \sum_{l=\langle N \rangle} a_{k-l} b_l = \sum_{l=2}^{4} a_{k-l} b_l = \sum_{l=2}^{4} a_{k-l} \cdot l$

$= 2a_{k-2} + 3a_{k-3} + 4a_{k-4}$

$c_3 = 2a_1 + 3a_0 + 4a_{-1} = 2a_1 + 3a_0 + 4a_7$

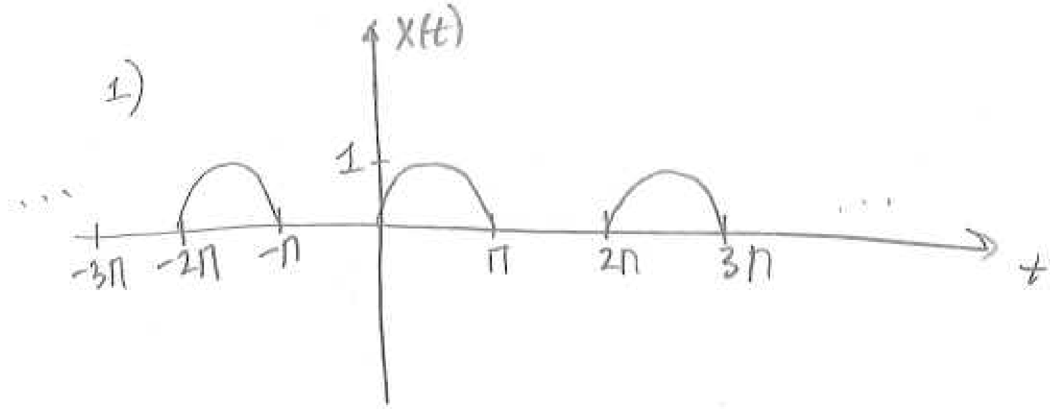
Question 2: [22%, Work-out question, Learning Objectives 1, 4, and 5] Consider the following signal:

$$x(t) = \begin{cases} \sin(t) & \text{if } 0 \le t \le \pi \\ 0 & \text{if } \pi \le t \le 2\pi \\ \text{periodic with period } T = 2\pi \end{cases}$$

and denote the corresponding Fourier series coefficients by a_k.

1. [4%] Plot $x(t)$ for the range of $-3\pi \le t \le 3\pi$.
2. [8%] Compute the value of a_0.
3. [10%] Compute the value of a_1.

1)



2) $a_0 = \frac{1}{T}\int_0^T x(t)\, e^{-j\omega t \cdot 0}\, dt = \frac{1}{2\pi}\int_0^\pi \sin t\, dt = \frac{-1}{2\pi}\cos t \Big|_0^\pi =$

$= -\frac{1}{2\pi}\left[\cos\pi - \cos 0\right] = \frac{2}{2\pi} = \frac{1}{\pi}$

3) $a_1 = \frac{1}{2\pi}\int_0^\pi \sin t\, e^{-j\omega t \cdot 1}\, dt = \frac{1}{2\pi}\int_0^\pi \sin t\, e^{-j\frac{2\pi}{2\pi}t}\, dt =$

$= \frac{1}{2\pi}\int_0^\pi \sin t\, e^{-jt}\, dt = \frac{1}{2\pi}\left(-\frac{jt}{2} - \frac{1}{4}e^{-j2t}\right)\Big|_0^\pi = \frac{1}{2\pi}\left[-\frac{j\pi}{2} - \frac{1}{4}e^{-j2\pi} + \frac{1}{4}\right] =$

$= \frac{1}{2\pi}\left(-\frac{j\pi}{2} - \frac{1}{4} + \frac{1}{4}\right) = -\frac{j}{4}$

$\boxed{e^{-j2\pi} = \cos(2\pi) - j\sin(2\pi) = 1}$

Question 3: [16%, Work-out question, Learning Objectives 3, 4, and 5] Consider the following signal:

$$x(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{if } 1 \leq t < 4 \\ \text{periodic with period } T = 4 & \end{cases}$$

and denote its Fourier series coefficients by a_k.

1. [6%] Assuming you know the values of a_k, plot $X(j\omega)$ for the range of $-1.1\pi \leq t \leq 1.1\pi$.

We then pass $x(t)$ through an ideal low-pass filter with cutoff frequency $\frac{\pi}{3}$ and denote the output as $y(t)$.

2. [10%] Plot $Y(j\omega)$ for the range of $-1.1\pi \leq t \leq 1.1\pi$.

   Hint 1: Your answer for this sub-question should not use a_k anymore. Namely, you may have to compute some a_k values for this sub-question. If your answer still contains some a_k values, then you will receive 8 points instead.

   Hint 2: If you do not know the expression of $X(j\omega)$ in the first sub-question, you can assume

$$X(j\omega) = \frac{\sin(3\omega)}{\omega}$$

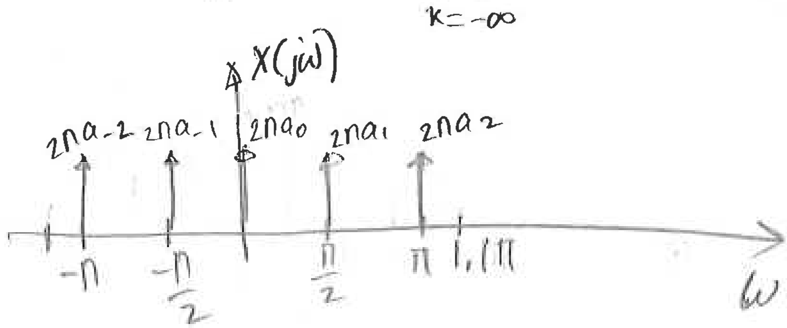
   and use it to plot $Y(j\omega)$. You will still receive full credit (10 points) if your answer is correct.

$$a_K = \frac{1}{T}\int_0^T x(t) e^{-j\omega kt}\, dt = \frac{1}{4}\int_0^4 x(t) e^{-j\frac{2\pi}{4}kt}\, dt = \frac{1}{4}\int_0^1 1 \cdot e^{-j\frac{\pi}{2}kt} =$$

$$= \frac{1}{4}\frac{1}{-j\frac{\pi}{2}K}\left[e^{-j\frac{\pi}{2}K} - 1\right] = \frac{1}{-j2\pi k} e^{-j\frac{\pi}{4}K}\left(e^{-j\frac{\pi}{4}K} - e^{j\frac{\pi}{4}k}\right) =$$

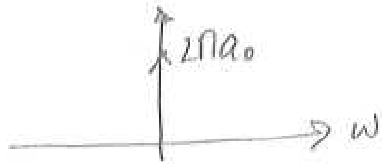
$$= \frac{1}{\pi k} e^{-j\frac{\pi}{4}K} \sin\left(\frac{\pi}{4}k\right)$$

1) $X(j\omega) = 2\pi \sum_{K=-\infty}^{\infty} a_K \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{\pi}{2}k)$



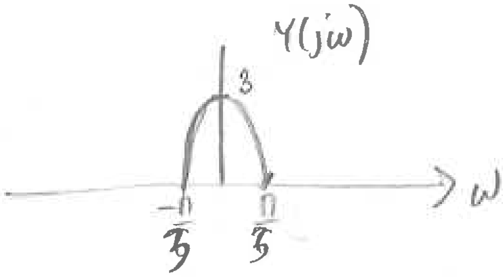
2)

$Y(j\omega)$



, where $a_0 = \lim_{k \to 0} \dfrac{\sin(\frac{\pi}{4}k)}{\pi k} = \dfrac{\pi/4}{\pi} = \dfrac{1}{4}$

if $X(j\omega) = \dfrac{\sin 3\omega}{\omega}$.



Question 4: [20%, Work-out question, Learning Objectives 3, 4, and 5] Consider an LTI system for which the input/output relationship is governed by the following differential equation.

$$y(t) + 2\frac{d}{dt}y(t) = 2x(t)$$

We also assume that the LTI system is *initially rest.* That is, if the input is $x(t) = 0$, then the output is $y(t) = 0$.

1. [8%] Find out the impulse response $h(t)$ of this system.
2. [12%] Find out the output $y(t)$ when the input is $x(t) = e^{-3(t-1)}\mathcal{U}(t-1)$.

Hint: If you do not know the $h(t)$ (or equivalently $H(j\omega)$), the answer to the first sub-question, you can assume $H(j\omega) = \frac{1}{(1+j\omega)^2}$. You will get full credit for the second sub-questions.

1) $Y(j\omega) + 2Y(j\omega)j\omega = 2X(j\omega)$

$Y(j\omega)(1+2j\omega) = 2X(j\omega)$

$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{1+2j\omega} = \frac{1}{\frac{1}{2}+j\omega}$

$h(t) = e^{-\frac{1}{2}t}u(t)$

2) $x(t) = e^{-3(t-1)}u(t)$

$e^{-3t}u(t) \Leftrightarrow \frac{1}{3+j\omega}$

$e^{-3(t-1)}u(t-1) \Leftrightarrow e^{-j\omega}\frac{1}{3+j\omega}$

$Y(j\omega) = e^{-j\omega}\frac{1}{3+j\omega}\cdot\frac{1}{\frac{1}{2}+j\omega} = \left(\frac{A}{3+j\omega} + \frac{B}{\frac{1}{2}+j\omega}\right)e^{-j\omega}$

$A(\frac{1}{2}+j\omega) + B(3+j\omega) = 1$

$\frac{1}{2}A + 3B = 1$ $\quad -\frac{1}{2}B + 3B = \frac{5}{2}B = 1$ $\quad B = \frac{2}{5}$ $\quad A = -\frac{2}{5}$

$A + B = 0$ $\quad y(t) = -\frac{2}{5}e^{-3(t-1)}u(t-1) + \frac{2}{5}e^{-\frac{1}{2}(t-1)}u(t-1)$

$A = -B$

$$H(j\omega) = \frac{1}{(1+j\omega)^2}$$

$$Y(j\omega) = \frac{1}{(1+j\omega)^2} \, \frac{1}{3+j\omega} \, e^{-j\omega} =$$

$$= e^{-j\omega} \left(\frac{A}{(1+j\omega)^2} + \frac{B}{1+j\omega} + \frac{C}{3+j\omega} \right)$$

$$A(3+j\omega) + B(1+j\omega)(3+j\omega) + C(1+j\omega)^2 = 1$$

$$3A + Aj\omega + B(3 + 4j\omega - \omega^2) + C(1 + 2j\omega - \omega^2) = 1$$

$$A + 4B + 2C = 0$$

$$-B - C = 0 \qquad B = -C \Rightarrow \begin{cases} A - 4C + 2C = 0 \\ 3A - 3C + C = 1 \end{cases}$$

$$3A + 3B + C = 1$$

$$A - 2C = 0$$

$$3A - 2C = 1 \qquad 2A = 1 \quad A = 1/2$$

$$C = 1/4 \qquad B = -1/4$$

$$y(t) = \frac{1}{2}(t-1)e^{-(t-1)}u(t-1) - \frac{1}{4}e^{-(t-1)}u(t-1) + \frac{1}{4}e^{-3(t-1)}u(t-1)$$

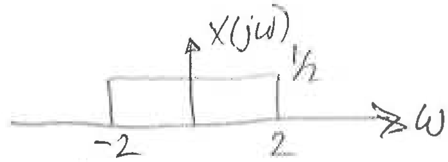
Question 5: [12%, Work-out question, Learning Objectives 3, 4, and 5]

Consider continuous-time signals $x(t) = \frac{\sin(2t)}{2\pi t}$ and $h(t) = \frac{\sin(2.5t)}{\pi t}$.

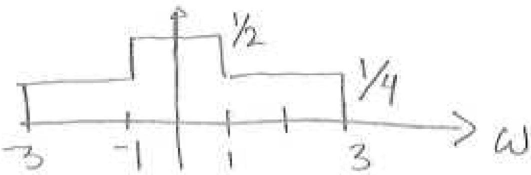
Define $y(t) = (x(t)\cos(t)) * h(t)$. That is, $y(t)$ is obtained by multiplying $x(t)$ by $\cos(t)$ and then passing it through an LTI system with impulse response $h(t)$.

Plot $Y(j\omega)$ for the range of $-4 \le \omega \le 4$.

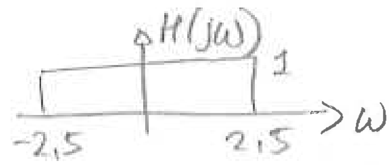
$x(t) = \frac{\sin(2t)}{2\pi t} \overset{FT}{\Longleftrightarrow}$



$x(t)\cdot\cos t \overset{FT}{\Longleftrightarrow}$



$h(t) \overset{FT}{\Longleftrightarrow}$



$(x(t)\cdot\cos t) * h(t) \Longleftrightarrow$