**Midterm #3 of ECE301, Prof. Wang's section**
6:30–7:30pm Wednesday, April 2, 2014, ME 1061,

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**

2. This is a closed book exam.

3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.

4. Use the back of each page for rough work.

5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

E-mail:

Signature:

*Question 1:* [30%, Work-out question, Learning Objectives 3, 4, and 5]

Consider a discrete-time signal

$$x[n] = \begin{cases} 2^{-n} & \text{if } -4 \leq n \leq 3 \\ \text{periodic with period } N = 8 \end{cases}$$

and let $a_k$ denote its Fourier series coefficients.

1. [8%] Compute the value of $a_4$.
2. [8%] Compute the value of $\sum_{k=0}^{7}(-1)^k a_k$.
3. [8%] Compute the value of $\sum_{k=0}^{7}|a_k|^2$.

We have another signal $y[n]$, for which the corresponding Fourier series coefficients are

$$b_k = \begin{cases} k & \text{if } 2 \leq k \leq 4 \\ 0 & \text{if } 0 \leq k \leq 1 \text{ or } 5 \leq k \leq 7 \end{cases}$$

Define $z[n] = x[n]y[n]$ and denote the corresponding Fourier series coefficients by $c_k$.

4. [6%] Write down the expression of $c_3$ in terms of $a_k$.

*Question 2:* [22%, Work-out question, Learning Objectives 1, 4, and 5] Consider the following signal:

$$x(t) = \begin{cases} \sin(t) & \text{if } 0 \leq t \leq \pi \\ 0 & \text{if } \pi \leq t \leq 2\pi \\ \text{periodic with period } T = 2\pi & \end{cases}$$

and denote the corresponding Fourier series coefficients by $a_k$.

1. [4%] Plot $x(t)$ for the range of $-3\pi \leq t \leq 3\pi$.
2. [8%] Compute the value of $a_0$.
3. [10%] Compute the value of $a_1$.

*Question 3:* [16%, Work-out question, Learning Objectives 3, 4, and 5] Consider the following signal:

$$x(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < 4 \\ \text{periodic with period } T = 4 \end{cases}$$

and denote its Fourier series coefficients by $a_k$.

1. [6%] Assuming you know the values of $a_k$, plot $X(j\omega)$ for the range of $-1.1\pi \le t \le 1.1\pi$.

We then pass $x(t)$ through an ideal low-pass filter with cutoff frequency $\frac{\pi}{3}$ and denote the output as $y(t)$.

2. [10%] Plot $Y(j\omega)$ for the range of $-1.1\pi \le t \le 1.1\pi$.

   Hint 1: Your answer for this sub-question should not use $a_k$ anymore. Namely, you may have to compute some $a_k$ values for this sub-question. If your answer still contains some $a_k$ values, then you will receive 8 points instead.

   Hint 2: If you do not know the expression of $X(j\omega)$ in the first sub-question, you can assume

   $$X(j\omega) = \frac{\sin(3\omega)}{\omega}$$

   and use it to plot $Y(j\omega)$. You will still receive full credit (10 points) if your answer is correct.

*Question 4:* [20%, Work-out question, Learning Objectives 3, 4, and 5] Consider an LTI system for which the input/output relationship is governed by the following differential equation.

$$y(t) + 2\frac{d}{dt}y(t) = 2x(t)$$

We also assume that the LTI system is *initially rest*. That is, if the input is $x(t) = 0$, then the output is $y(t) = 0$.

1. [8%] Find out the impulse response $h(t)$ of this system.

2. [12%] Find out the output $y(t)$ when the input is $x(t) = e^{-3(t-1)}\mathcal{U}(t-1)$.

Hint: If you do not know the $h(t)$ (or equivalently $H(j\omega)$), the answer to the first sub-question, you can assume $H(j\omega) = \frac{1}{(1+j\omega)^2}$. You will get full credit for the second sub-questions.

*Question 5:* [12%, Work-out question, Learning Objectives 3, 4, and 5]

Consider continuous-time signals $x(t) = \frac{\sin(2t)}{2\pi t}$ and $h(t) = \frac{\sin(2.5t)}{\pi t}$.

Define $y(t) = (x(t)\cos(t)) * h(t)$. That is, $y(t)$ is obtained by multiplying $x(t)$ by $\cos(t)$ and then passing it through an LTI system with impulse response $h(t)$.

Plot $Y(j\omega)$ for the range of $-4 \leq \omega \leq 4$.

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(j\omega)e^{j\omega n}d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi}e^{\sigma t}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{10}$$

Z transform

$$x[n] = r^n\mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{12}$$

**TABLE 3.1** PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

| Property | Section | Periodic Signal | Fourier Series Coefficients |
|---|---|---|---|
| | | $\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$ | $a_k$ <br> $b_k$ |
| Linearity | 3.5.1 | $Ax(t) + By(t)$ | $Aa_k + Bb_k$ |
| Time Shifting | 3.5.2 | $x(t - t_0)$ | $a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$ |
| Frequency Shifting | | $e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$ | $a_{k-M}$ |
| Conjugation | 3.5.6 | $x^*(t)$ | $a^*_{-k}$ |
| Time Reversal | 3.5.3 | $x(-t)$ | $a_{-k}$ |
| Time Scaling | 3.5.4 | $x(\alpha t), \alpha > 0$ (periodic with period $T/\alpha$) | $a_k$ |
| Periodic Convolution | | $\int_T x(\tau)y(t - \tau)d\tau$ | $Ta_k b_k$ |
| Multiplication | 3.5.5 | $x(t)y(t)$ | $\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$ |
| Differentiation | | $\frac{dx(t)}{dt}$ | $jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$ |
| Integration | | $\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$) | $\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$ |
| Conjugate Symmetry for Real Signals | 3.5.6 | $x(t)$ real | $\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | 3.5.6 | $x(t)$ real and even | $a_k$ real and even |
| Real and Odd Signals | 3.5.6 | $x(t)$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | | $\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$ | $\mathcal{Re}\{a_k\}$ <br> $j\mathcal{Im}\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

## Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

## 3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

**TABLE 3.2** PROPERTIES OF DISCRETE-TIME FOURIER SERIES

| Property | Periodic Signal | Fourier Series Coefficients |
|---|---|---|
| | $\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period $N$ and fundamental frequency $\omega_0 = 2\pi/N$ | $\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period $N$ |
| Linearity | $Ax[n] + By[n]$ | $Aa_k + Bb_k$ |
| Time Shifting | $x[n - n_0]$ | $a_k e^{-jk(2\pi/N)n_0}$ |
| Frequency Shifting | $e^{jM(2\pi/N)n} x[n]$ | $a_{k-M}$ |
| Conjugation | $x^*[n]$ | $a^*_{-k}$ |
| Time Reversal | $x[-n]$ | $a_{-k}$ |
| Time Scaling | $x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period $mN$) | $\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$ |
| Periodic Convolution | $\sum_{r=\langle N\rangle} x[r]y[n-r]$ | $Na_k b_k$ |
| Multiplication | $x[n]y[n]$ | $\sum_{l=\langle N\rangle} a_l b_{k-l}$ |
| First Difference | $x[n] - x[n-1]$ | $(1 - e^{-jk(2\pi/N)})a_k$ |
| Running Sum | $\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$ | $\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$ |
| Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | $x[n]$ real and even | $a_k$ real and even |
| Real and Odd Signals | $x[n]$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | $\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$ | $\mathcal{R}e\{a_k\}$<br>$j\mathcal{I}m\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N\rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k \rvert^2$$

## 4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

**TABLE 4.1** PROPERTIES OF THE FOURIER TRANSFORM

| Section | Property | Aperiodic signal | Fourier transform |
|---|---|---|---|
| | | $x(t)$<br>$y(t)$ | $X(j\omega)$<br>$Y(j\omega)$ |
| 4.3.1 | Linearity | $ax(t) + by(t)$ | $aX(j\omega) + bY(j\omega)$ |
| 4.3.2 | Time Shifting | $x(t - t_0)$ | $e^{-j\omega t_0}X(j\omega)$ |
| 4.3.6 | Frequency Shifting | $e^{j\omega_0 t}x(t)$ | $X(j(\omega - \omega_0))$ |
| 4.3.3 | Conjugation | $x^*(t)$ | $X^*(-j\omega)$ |
| 4.3.5 | Time Reversal | $x(-t)$ | $X(-j\omega)$ |
| 4.3.5 | Time and Frequency Scaling | $x(at)$ | $\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$ |
| 4.4 | Convolution | $x(t) * y(t)$ | $X(j\omega)Y(j\omega)$ |
| 4.5 | Multiplication | $x(t)y(t)$ | $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$ |
| 4.3.4 | Differentiation in Time | $\frac{d}{dt}x(t)$ | $j\omega X(j\omega)$ |
| 4.3.4 | Integration | $\int_{-\infty}^{t} x(t)dt$ | $\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$ |
| 4.3.6 | Differentiation in Frequency | $tx(t)$ | $j\frac{d}{d\omega}X(j\omega)$ |
| 4.3.3 | Conjugate Symmetry for Real Signals | $x(t)$ real | $\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$ |
| 4.3.3 | Symmetry for Real and Even Signals | $x(t)$ real and even | $X(j\omega)$ real and even |
| 4.3.3 | Symmetry for Real and Odd Signals | $x(t)$ real and odd | $X(j\omega)$ purely imaginary and odd |
| 4.3.3 | Even-Odd Decomposition for Real Signals | $x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]<br>$x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real] | $\Re e\{X(j\omega)\}$<br>$j\Im m\{X(j\omega)\}$ |

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

**TABLE 4.2** BASIC FOURIER TRANSFORM PAIRS

| Signal | Fourier transform | Fourier series coefficients (if periodic) |
|---|---|---|
| $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$ | $a_k$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ | $a_1 = 1$<br>$a_k = 0$, otherwise |
| $\cos \omega_0 t$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | $a_1 = a_{-1} = \frac{1}{2}$<br>$a_k = 0$, otherwise |
| $\sin \omega_0 t$ | $\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | $a_1 = -a_{-1} = \frac{1}{2j}$<br>$a_k = 0$, otherwise |
| $x(t) = 1$ | $2\pi\delta(\omega)$ | $a_0 = 1$, $a_k = 0$, $k \neq 0$<br>(this is the Fourier series representation for any choice of $T > 0$) |
| Periodic square wave<br>$x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \leq \frac{T}{2} \end{cases}$<br>and<br>$x(t+T) = x(t)$ | $\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$ | $\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$ |
| $\sum_{n=-\infty}^{+\infty} \delta(t - nT)$ | $\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ | $a_k = \frac{1}{T}$ for all $k$ |
| $x(t) \begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$ | $\frac{2 \sin \omega T_1}{\omega}$ | — |
| $\frac{\sin Wt}{\pi t}$ | $X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$ | — |
| $\delta(t)$ | $1$ | — |
| $u(t)$ | $\frac{1}{j\omega} + \pi\delta(\omega)$ | — |
| $\delta(t - t_0)$ | $e^{-j\omega t_0}$ | — |
| $e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$ | $\frac{1}{a + j\omega}$ | — |
| $te^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$ | $\frac{1}{(a + j\omega)^2}$ | — |
| $\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$,<br>$\mathcal{R}e\{a\} > 0$ | $\frac{1}{(a + j\omega)^n}$ | — |