Question 1: [24%, Work-out question, Learning Objectives 1, 2, and 3] Consider two continuous-time signals $x(t)$ and $y(t)$.

$$x(t) = \mathcal{U}(t-2) - \mathcal{U}(t+2) + \delta(t)$$

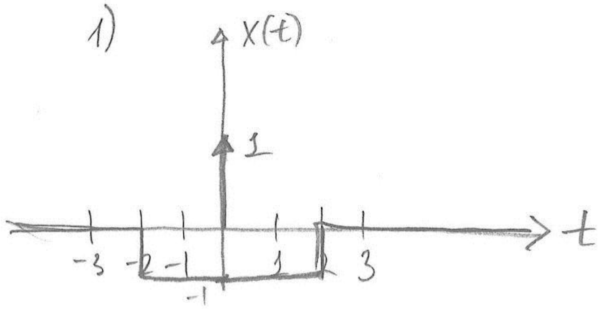
$$h(t) = e^{-t}\mathcal{U}(t-1) = \begin{cases} e^{-t} & \text{if } t \geq 1 \\ 0 & \text{if } t < 1 \end{cases} = e^{-t} \cdot u(t-1)$$

1. [4%] Plot $x(t)$ for the range of $-3 \leq t \leq 3$.
2. [20%] Compute the expression of

$$y(t) = x(t) * h(t).$$

Hint: You may want to express $x(t)$ as $x(t) = x_1(t) + x_2(t)$, compute the corresponding output $y_1(t)$ and $y_2(t)$ separately, and then assemble the final answer.

1)



2) $x(t) = x_1(t) + x_2(t)$, $\quad x_1(t) = \delta(t)$, $\quad x_2(t) = -u(t+2) + u(t-2)$

$$y_1(t) = \int_{-\infty}^{\infty} x_1(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau) e^{-(t-\tau)} u(t-\tau-1) d\tau$$

$$= e^{-t} u(t-1)$$

$$y_2(t) = \int_{-\infty}^{\infty} (u(\tau-2) - u(\tau+2)) \cdot e^{-(t-\tau)} u(t-\tau-1) d\tau =$$

$$= \int_{-\infty}^{\infty} e^{-(t-\tau)} u(\tau-2) u(t-\tau-1) d\tau - \int_{-\infty}^{\infty} e^{-(t-\tau)} u(\tau+2) u(t-\tau-1) d\tau$$

$$= e^{-t} \int_{2}^{\infty} e^{\tau} u(t-\tau-1) d\tau - e^{-t} \int_{-2}^{\infty} e^{\tau} u(t-\tau-1) d\tau =$$

$$= e^{-t} \left(\int_{2}^{t-1} e^{\tau} d\tau \right) \cdot u(t-1-2) - e^{-t} \left(\int_{-2}^{t-1} e^{\tau} d\tau \right) \cdot u(t-1+2) =$$

$$= e^{-t} u(t-3) \left[e^{t-1} - e^{2} \right] - e^{-t} u(t+1) \left[e^{t-1} - e^{-2} \right]$$

$$= e^{-1} u(t-3) - e^{-t+2} u(t-3) - e^{-1} u(t+1) + e^{-(t+2)} u(t+1)$$

$$y(t) = y_1(t) + y_2(t) = e^{-t} u(t-1) + e^{-1} \left[u(t-3) - u(t+1) \right]$$
$$- e^{-t+2} u(t-3) + e^{-(t+2)} u(t+1)$$

Question 2: [16%, Work-out question, Learning Objectives 1, 2, and 3] Consider the following system:

$$y[n] = x[n]e^{x[n-1]}$$

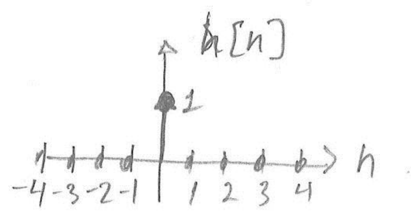
1. [4%] Plot the impulse response $h[n]$ for the range of $-4 \leq n \leq 4$.
2. [4%] When the input is

$$x[n] = \begin{cases} 1 & \text{if } -1 \leq n \leq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

   Plot the output signal $y[n]$ for the range of $-4 \leq n \leq 4$.

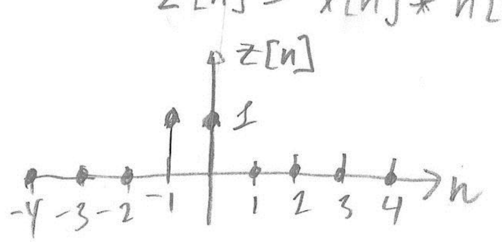
3. [4%] Continue from the previous question. Plot the signal $z[n] = x[n] * h[n]$.
4. [4%] Is $y[n]$ identical to $z[n]$? Write one sentence or two to justify your answer for this yes/no question.

1). $h[n] = \delta[n] e^{\delta[n-1]} = \begin{cases} 1 \cdot e^0, & n=0 \\ 0 \cdot e^1, & n=1 \\ 0, & ow \end{cases} = \begin{cases} 1, & n=0 \\ 0, & ow \end{cases} = \delta[n]$



3) $x[n] = \delta[n+1] + \delta[n]$

$z[n] = x[n] * h[n] = x[n] * \delta[n] = x[n]$



4) $y[n] \neq z[n]$
The sys is not LTI

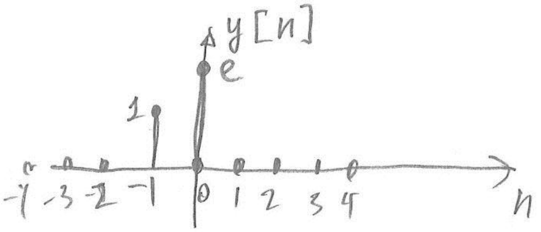
2) $y[n] = 0$ for $n < -1$

$y[-1] = x[-1] e^{x[-2]} = 1$

$y[0] = x[0] e^{x[-1]} = 1 \cdot e^1 = e$

$y[1] = x[1] e^{x[0]} = 0$

$y(n) = 0$ for $n > 1$



Question 3: [24%, Work-out question, Learning Objectives 1, 2, 3, and 5] Consider the following LTI system

$$y(t) = \int_{s=t-2}^{t} x(s)e^{-(t-s)}ds.$$

1. [8%] Compute the impulse response $h(t)$.
2. [16%] Compute the output $y(t)$ when the input is $x(t) = e^{-jt} + e^{j(\pi t + \frac{\pi}{3})}$. (Hint: If you do not know the answer of the first sub-question, you can assume that $h(t) = e^{-|t|}$. You will still get full credit if your answer is correct.)

1) $h(t) = \int_{t-2}^{t} \delta(s)e^{-(t-s)}ds = e^{-t}$, for $t \geq 0$ and $t-2 \leq 0$

$\Rightarrow \quad 0 \leq t \leq 2$

$h(t) = e^{-t}(u(t) - u(t-2))$

2) $y(t) = x(t) * h(t)$.

$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-t}(u(t)-u(t-2))e^{-j\omega t}dt =$

$= \int_{0}^{2} e^{-t(1+j\omega)}dt = \frac{-1}{1+j\omega}\left[e^{-2(1+j\omega)} - 1\right] = \frac{1-e^{-2(1+j\omega)}}{1+j\omega}$

$e^{j\omega t} \rightarrow \boxed{LTI} \rightarrow H(j\omega)e^{j\omega t}$

$y(t) = e^{-jt} \cdot H(j\cdot(-1)) + e^{j\frac{\pi}{3}} \cdot e^{j\pi t} \cdot H(j\pi) =$

$= e^{-jt} \cdot \frac{1-e^{-2(1-j)}}{1-j} + e^{j(\frac{\pi}{3}+\pi t)} \frac{1-e^{-2(1+j\pi)}}{1+j\pi}$

for $h(t) = e^{-|t|}$ $\qquad H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt =$

$$= \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt + \int_{-\infty}^{0} e^{t} e^{-j\omega t}\, dt = \frac{-1}{1+j\omega}\left[e^{-(1+j\omega)\infty} - 1\right]$$

$$+ \frac{1}{1-j\omega}\left[e^{0(1-j\omega)} - e^{-\infty}\right] = \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{2}{1+\omega^2}$$

$$y(t) = e^{-jt} \cdot \frac{2}{1+(-1)^2} + e^{j(\pi t + \frac{\pi}{3})} \cdot \frac{2}{1+\pi^2} =$$

$$= e^{-jt} + \frac{2}{1+\pi^2} e^{j(\pi t + \frac{\pi}{3})}$$

Question 4: [16%, Work-out question, Learning Objectives 4 and 5] Consider the following signal.

$$x(t) = \cos(\frac{4\pi}{3}t) + \sin(5\pi t) \tag{2}$$

Compute the Fourier series representation of $x(t)$.

$$x(t) = \frac{e^{j\frac{4\pi}{3}t} + e^{-j\frac{4\pi}{3}t}}{2} + \frac{e^{j5\pi t} - e^{-j5\pi t}}{2j}$$

the period of $\cos(\frac{4\pi}{3}t)$ $\quad \frac{4\pi}{3}T_1 = 2\pi k \quad 2T_1 = 3k \quad T_1 = \frac{3}{2}$

the period of $\sin(5\pi t)$ $\quad 5\pi T_2 = 2\pi k \quad T_2 = \frac{2}{5}$

$T = LCM(\frac{3}{2}, \frac{2}{5}) = 6$, $\quad w = \frac{2\pi}{6} = \frac{\pi}{3}$.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jkwt} = \frac{1}{2} e^{j\frac{\pi}{3}\cdot 4t} + \frac{1}{2} e^{-j\frac{\pi}{3}\cdot 4t} + \frac{1}{2j} e^{j\frac{\pi}{3}\cdot 15t} - \frac{1}{2j} e^{-j\frac{\pi}{3}\cdot 15t}$$

$$\Rightarrow \quad a_k = \begin{cases} \frac{1}{2}, & k = -4, 4 \\ \frac{1}{2j}, & k = 15 \\ -\frac{1}{2j}, & k = -15 \\ 0, & ow \end{cases}$$

Question 5: [20%, Multiple Choices] The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \int_{-\infty}^{2t} x_1(s)ds. \tag{3}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = \begin{cases} x_2[n/5] & \text{if } \frac{n}{5} \text{ is an integer} \\ y_2[n-1] & \text{if } \frac{n}{5} \text{ is not an integer} \end{cases} \tag{4}$$

Answer the following questions

1. [4%, Learning Objective 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Learning Objective 1] Is System 1 causal? Is System 2 causal?
3. [4%, Learning Objective 1] Is System 1 stable? Is System 2 stable?
4. [4%, Learning Objective 1] Is System 1 linear? Is System 2 linear?
5. [4%, Learning Objective 1] Is System 1 time-invariant? Is System 2 time-invariant?

1. NO
NO
unstable
linear
time-varying

2. NO
NO
STABLE
linear
time varying