Midterm #3 of ECE301, Section 1 and Section 2
6:30-7:30pm, Wednesday, November 12, 2014.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solution Key

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date: 11-12-2014

Question 1: [28%, Work-out question, Learning Objectives 2, 3, 4, 5]

Consider an input signal $x(t)$ and an LTI system with impulse response $h(t)$. The expressions of $x(t)$ and $h(t)$ are:

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$$

$$h(t) = \begin{cases} e^{-j2\pi t} & \text{if } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

1. [10%] Find the frequency response $H(j\omega)$ of the LTI system and plot it for the range of $-4\pi \le \omega \le 4\pi$.
2. [8%] Find the Fourier transform $X(j\omega)$ of the input $x(t)$ and plot it for the range of $-4\pi \le \omega \le 4\pi$.
3. [10%] Let $y(t)$ denote the output of the LTI system when the input is $x(t)$. Do (i) find the expression of $y(t)$; and (ii) plot $Y(j\omega)$ for the range of $-4\pi \le \omega \le 4\pi$.

   Hint: If you do not know the answers to the first two subquestions, you can assume $y(t) = (\cos(\pi t + 1) + \sin(2\pi t) + 3) * (\mathcal{U}(t+1) - \mathcal{U}(t-1))$ and use this $y(t)$ to answer tasks (i) and (ii). You will get 10 points if your answers are correct.

1/ Let $x_p(t) = \begin{cases} 1 & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$

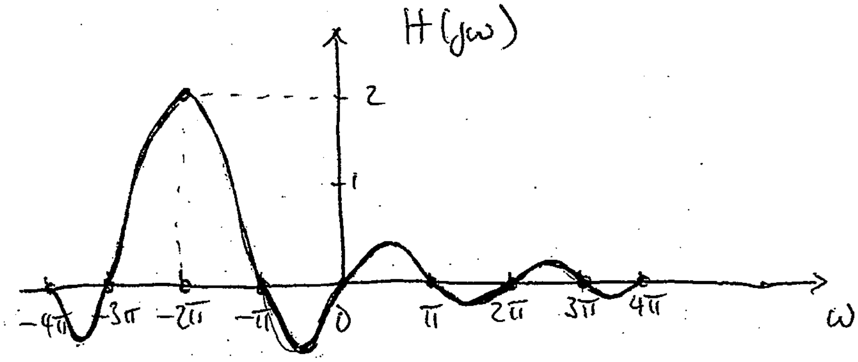
then $h(t) = e^{-j2\pi t} \cdot x_p(t)$

thus $H(j\omega) = \frac{1}{2\pi}\left[2\pi\,\delta(\omega + 2\pi) * X_p(j\omega)\right]$ by Multiplication property

where $e^{-j2\pi t} \xrightarrow{F.T} 2\pi\,\delta(\omega + 2\pi)$

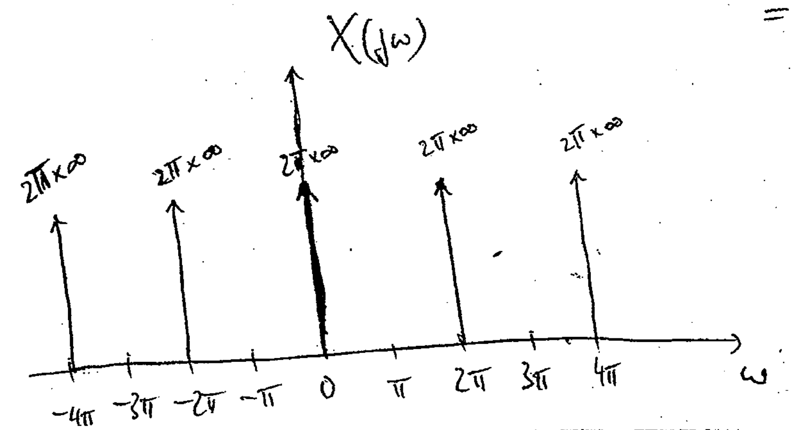
and $X_p(j\omega) = 2\,\frac{\sin(\omega \cdot 1)}{\omega} = \frac{2\sin(\omega)}{\omega}$

$\Rightarrow H(j\omega) = X_p[j(\omega + 2\pi)] = 2\,\frac{\sin(\omega + 2\pi)}{(\omega + 2\pi)}$



2/ $$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-k) \xrightarrow{F.T} \frac{2\pi}{1} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{1}\right)$$

$$= \underline{X(j\omega)}$$



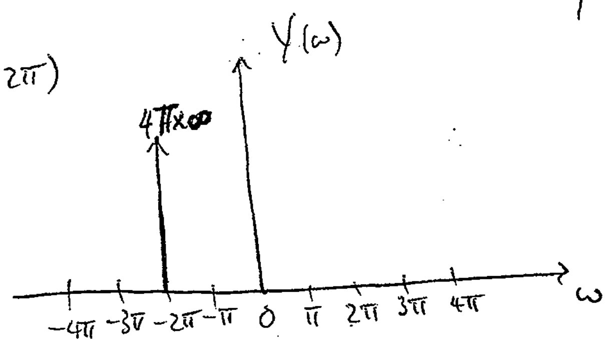
3/ i/ $Y(\omega) = X(\omega) \cdot H(\omega)$

$$Y(\omega) = 2 \cdot 2\pi \ \delta(\omega + 2\pi)$$

$$Y(\omega) = 4\pi \ \delta(\omega + 2\pi)$$

thus $y(t) = \mathcal{F}^{-1}\{Y(\omega)\} = 2e^{-j2\pi t}$ Using the Fourier Transform pairs

ii/ $Y(\omega) = 2.2\pi \ \delta(\omega + 2\pi)$



Question 2: [22%, Work-out question, Learning Objectives 4, 5] Consider a periodic discrete time signal $x[n]$, of which the period is 50. Suppose we also know that the Discrete-Time Fourier Series Coefficients $x[n]$ is

$$a_k = \begin{cases} 1 & \text{if } 1 \le k \le 19 \\ -1 & \text{if } 31 \le k \le 49 \\ 0 & \text{if } k = 0 \text{ or } 20 \le k \le 30 \end{cases}$$

1. [7%] What is value of $\sum_{n=51}^{100} x[n]$?
2. [7%] What is the value of $x[-25]$?
3. [8%] What is the average power of the signal $x[n]$?

1/ From Discrete time Fourier Series formula, we have

that $a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}$

thus $a_0 = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \cdot 0} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]$

$\Rightarrow \frac{1}{50} \sum_{n=51}^{100} x[n] = a_0 \Rightarrow \sum_{n=51}^{100} x[n] = 50 a_0$

since $a_0 = 0$ as given in the problem,

We get : $\sum_{n=51}^{100} x[n] = \underline{\underline{0}}$

2/ $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}$

thus $x[-25] = \sum_{k=0}^{49} a_k e^{jk\frac{2\pi}{50}(-25)} = \sum_{k=0}^{49} a_k e^{-jk\pi}$

$$\Rightarrow \quad x[-25] = \sum_{k=1}^{19} 1 \cdot e^{-jk\pi} + \sum_{k=31}^{49} (-1) \cdot e^{-jk\pi}$$

$$= \sum_{k=1}^{19} \left(e^{-j\pi}\right)^k + \sum_{k=31}^{49} \left(e^{-j\pi}\right)^k$$

$$= \frac{e^{-j\pi} - \left(e^{-j\pi}\right)^{20}}{1 - e^{-j\pi}} - \frac{\left(e^{-j\pi}\right)^{31} - \left(e^{-j\pi}\right)^{50}}{1 - e^{-j\pi}}$$

$$= \frac{e^{-j\pi} - e^{-j20\pi} - e^{-j31\pi} + e^{-j50\pi}}{1 - e^{-j\pi}} = \frac{-1 - 1 - (-1) + 1}{1 - (-1)}$$

$$x[-25] = \frac{-2+2}{2} = \underline{\underline{0}}$$

3/ Avg Power $= \frac{1}{N} \sum_{n \in \langle N \rangle} |x[n]|^2$

By Parserval's Theorem, we have $\frac{1}{N} \sum_{n = \langle N \rangle} |x[n]|^2 = \sum_{k = \langle N \rangle} |a_k|^2$

$$\Rightarrow \quad \text{Avg Power} = \frac{1}{50} \sum_{n=0}^{49} |x[n]|^2 = \sum_{k=0}^{49} |a_k|^2$$

$$\text{Avg Power} = \sum_{k=1}^{19} (1)^2 + \sum_{k=31}^{49} (-1)^2$$

$$= 19 + 19 = \underline{\underline{38}}$$

Question 3: [12%, Work-out question, Learning Objectives 4, 5]
Consider a discrete-time signal:

$$x[n] = \begin{cases} \cos(n) & \text{if } -10 \leq n \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

1. [10%] Find the Discrete-Time Fourier Transform $X(e^{j\omega})$ of $x[n]$.
2. [2%] Is your $X(e^{j\omega})$ periodic?

Hint: You may need to use the formula $\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}$ if $r \neq 1$.

1/ $X(e^{j\omega}) = \sum_{n=-10}^{10} \cos n \, e^{-j\omega n} = \sum_{n=-10}^{-1} \cos n \, e^{-j\omega n} + 1 + \sum_{n=1}^{10} \cos n \, e^{-j\omega n}$

$X(e^{j\omega}) = \sum_{n=1}^{10} \cos n \, e^{j\omega n} + 1 + \sum_{n=1}^{10} \cos n \, e^{-j\omega n}$

Using Euler's, we have $\cos n = \frac{1}{2} e^{jn} + \frac{1}{2} e^{-jn}$

$\Rightarrow X(e^{j\omega}) = \frac{1}{2} \sum_{n=1}^{10} e^{jn} \cdot e^{j\omega n} + \frac{1}{2} \sum_{n=1}^{10} e^{-jn} \cdot e^{j\omega n} + 1 + \frac{1}{2} \sum_{n=1}^{10} e^{jn} e^{-j\omega n} + \frac{1}{2} \sum_{n=1}^{10} e^{-jn} e^{-j\omega n}$

$= 1 + \frac{1}{2} \sum_{n=1}^{10} e^{j(1+\omega)n} + \frac{1}{2} \sum_{n=1}^{10} e^{-j(1-\omega)n} + \frac{1}{2} \sum_{n=1}^{10} e^{j(1-\omega)n} + \frac{1}{2} \sum_{n=1}^{10} e^{-j(1+\omega)n}$

$= 1 + \frac{1}{2} \left[\frac{e^{j(1+\omega)} - e^{10j(1+\omega)}}{1 - e^{j(1+\omega)}} + \frac{e^{-j(1-\omega)} - e^{-10j(1-\omega)}}{1 - e^{-j(1-\omega)}} + \frac{e^{j(1-\omega)} - e^{10j(1-\omega)}}{1 - e^{j(1-\omega)}} + \frac{e^{-j(1+\omega)} - e^{-10j(1+\omega)}}{1 - e^{-j(1+\omega)}} \right]$

2/ Yes, $X(e^{j\omega})$ is periodic.

Question 4: [15%, Work-out question, Learning Objectives 3, 4, 5] Consider the following signal

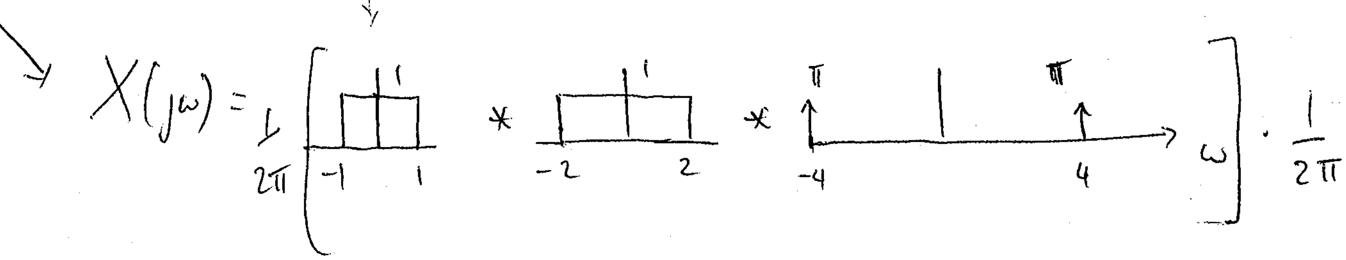
$$x(t) = \frac{\sin(t)\cos(4t)\sin(2t)}{\pi^2 t^2}. \tag{1}$$

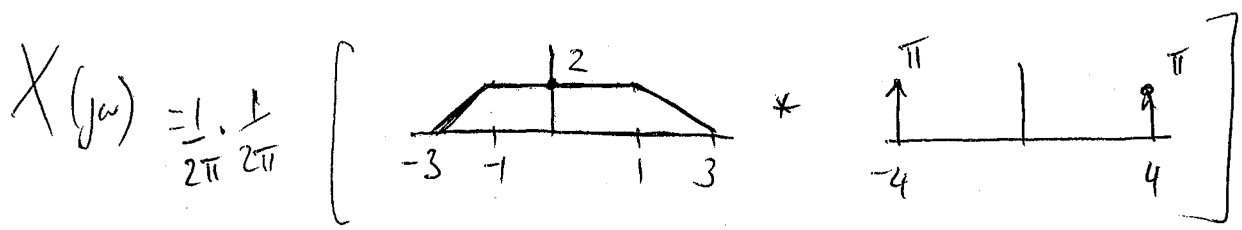
Plot the Continuous-Time Fourier transform $X(j\omega)$ for the range of $-8 < \omega < 8$.

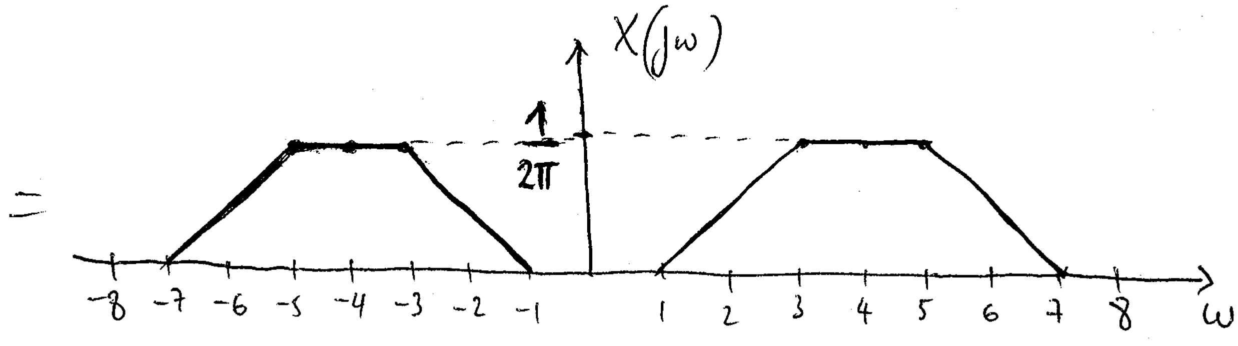
Hint: If you do not know the answer, you should write down as much as you know so that partial credit can be given.

$$x(t) = \frac{\sin(t)}{\pi t} \cdot \frac{\sin(2t)}{\pi t} \cdot \cos(4t)$$

F.T







Question 5: [23%, Work-out question, Learning Objective 3, 4, 5]

An AM radio station A would like to send an input signal $x_1(t)$ over a sine carrier of frequency 920k Hz. More specifically, we denote the input signal as $x_1(t)$ and use $y(t)$ to denote the AM modulated signal, which will be sent out by the transmitter.

1. [2%] What is the value of the carrier frequency with the unit being (rad/sec)?
2. [2%] Write down the input/output relationship (equation) between $x_1(t)$ and $y(t)$. Hint: Please pay attention to the requirement that the transmitter is using a sine signal as the carrier, not a cosine signal.

A nearby AM radio station B would like to send another signal $x_2(t)$ over a sine carrier but this time the carrier frequency is 930k Hz. It turns out that when both stations A and B are transmitting simultaneously, their signals are "interfering heavily with each other". It turns out that the reason is that both radio stations A and B forgot to low-pass filter their signals $x_1(t)$ and $x_2(t)$ before their transmission.

3. [5%] Explain why "not low-pass filtering it before transmission" can cause problems when both radio stations transmit simultaneously. (A very quick sentence or two will suffice).

To avoid interference, it is critical that we low-pass filter the base-band signals $x_1(t)$ and $x_2(t)$ before transmitting at carrier frequencies 920k and 930k Hz, respectively.

4. [8%] What is the (largest) cutoff frequency of the low-pass filter one should use to avoid the interference between radio stations A and B? Please make sure your unit is correct.
5. [6%] The receiver 1 uses synchronous demodulation to listen to radio station A. Let $w(t)$ denote the resulting signal after demodulation. Write down the relationship between $y(t)$ and $w(t)$.

   Hint: Your answer should consist of statements like "multiplying ....." and/or "using a filter....." Please be specific about the parameters of the filters. If you prefer, you can also use a block diagram (flow chart) to describe your demodulation system instead of using sentences.

1/ $f_c = 920\text{ kHz} \Rightarrow w_c = 2\pi f_c = 2\pi \times 920000 \text{ rad/s}$

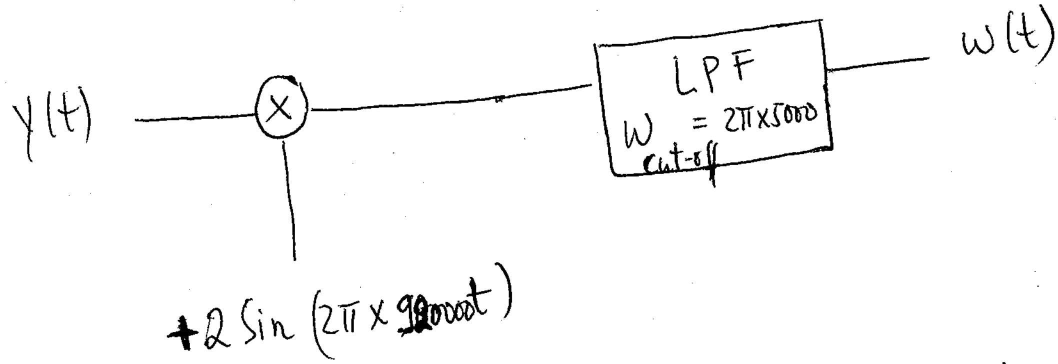
2/ $y(t) = x_1(t) \times \cancel{\sin(2\pi \times 920t)}$ $\sin(2\pi \times 920000t)$

3/ If both radio signals have bandwidth greater than 10 kHz, the modulated signals will interfere unless a low pass filter is used to remove frequency content above 5 kHz at baseband.

4| max $W_{cut\text{-}off} = 5000 \times 2\pi$ rad/s

or $f_{cut\text{-}off} = 5$ kHz

5|



where $w(t) = x_1(t)$