**Final Exam of ECE301, Section 1 and Section 2**
10:30am–12:30pm, Friday, December 19, 2014, EE 129.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**

2. This is a closed book exam.

3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.

4. Use the back of each page for rough work.

5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature:                                    Date:

*Question 1:* [17%, Work-out question]

   1. [1%] What does the acronym "AM-SSB" stand for?

Prof. Wang wanted to transmit an AM-SSB signal. To that end, he wrote the following MATLAB code.

```
% Initialialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read two different .wav files
[x1, f_sample, N]=wavread('x1');
x1=x1';
[x2, f_sample, N]=wavread('x2');
x2=x2';

% Step 0: Initialize several parameters
W_1=pi*4000;
W_2=pi*6000;
W_3=pi*12000;
W_4=pi*6000;
W_5=pi*2000;
W_6=?????;
W_7=????;


% Step 1: Make the signals band-limited.
h=1/(pi*t).*(sin(W_1*t));
x1_new=ece301conv(x1, h);
x2_new=ece301conv(x2, h);

% Step 2: Multiply x1_new and x2_new with a cosine wave.
x1_h=x1_new.*cos(W_2*t);
x2_h=x2_new.*cos(W_3*t);

% Step 3: Keep one of the two side bands
h_one=1/(pi*t).*(sin(W_4*t))-1/(pi*t).*(sin(W_5*t));
h_two=1/(pi*t).*(sin(W_6*t))-1/(pi*t).*(sin(W_7*t));
```

```
x1_sb=ece301conv(x1_h, h_one);
x2_sb=ece301conv(x2_h, h_two);

% Step 4: Create the transmitted signal
y=x1_sb+x2_sb;
wavwrite(y', f_sample, N, 'y.wav');
```

2. [1.5%] What is the bandwidth (Hz) of the signal x1_new?

3. [2.5%] Is this AM-SSB transmitting an upper-side-band signal or a lower-side-band signal?

4. [4%] What should the values of W_6 and W_7 be in the MATLAB code?

Knowing that Prof. Wang used the above code to generate the "y.wav" file, a student tried to demodulate the output waveform "y.wav" by the following code.

```
% Initialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read the .wav files
[y, f_sample, N]=wavread('y');
y=y';

% Initialize several parameters
W_8=????;
W_9=????;
W_10=????;

% Create the low-pass filter.
h_M=1/(pi*t).*(sin(W_8*t));

% demodulate signal 1
y1=4*y.*cos(W_9*t);
x1_hat=ece301conv(y1,h_M);

sound(x1_hat,f_sample)

% demodulate signal 2
y2=4*y.*cos(W_10*t);
x2_hat=ece301conv(y2,h_M);

sound(x2_hat,f_sample)
```

5. [3%] Continue from the previous question. What should the values of W_8 to W_10 in the MATLAB code?

6. [5%] It turns out that using the above MATLAB code, we can hear the sound properly when playing "sound(x2_hat,f_sample)" but there is some problem when playing "sound(x1_hat,f_sample)". Please (i) Describe why there is a problem when playing "sound(x1_hat,f_sample)" and (ii) Describe how we can fix the code so that we can hear x1_hat propertly.

Hint: If you do not know the answers of Q1.2 to Q1.6, please simply draw the AMSSB modulation and demodulation diagrams and mark carefully all the parameter values. You will receive 9 points for Q1.2 to Q1.6.

*Question 2:* [16%, Work-out question]

Consider a continuous time signal $x(t) = \sin(2\pi \cdot 500 \cdot t)$ and we use a digital voice recorder to convert the continuous time signal $x(t)$ to its discrete time counter part $x[n]$. Each $x[n]$ is represented by 16 bits (or equivalently 2 bytes). The digital voice recorder we use has a setting that allows you to change its sampling frequency.

Answer the following question.

1. [1%] Suppose we set the sampling frequency to 1000 Hz. What is the sampling period $T$?

2. [1%] Continue from the previous question. What is the relationship between $x(t)$ and $x(n)$?

3. [1.5%] Continue from the previous question. Suppose we start our recording at time $t = -1$ and the recording lasts for for 4 seconds, i.e., from $t = -1$ to 3. What is the file size after we finish recording? Please pay attention to the unit of your answer.

4. [2.5%] Continue from the previous question. Plot $x[n]$ for the range of $n = -3$ to 3.

5. [2.5%] Continue from the previous question. We use "zero-order hold" to reconstruct the original signal. We denote the output by $\hat{x}_{\text{ZOH}}(t)$. Plot $\hat{x}_{\text{ZOH}}(t)$ for the range of $t = -1$ to 3.

   [Alternative question:] If you do not know the answer to this question, you can assume that the sampling period is $T = 1/3$ and the sampled values are $x[n] = \delta[n] + 2\delta[n-4]$. Plot the Zero-Order-Hold output $\hat{x}_{\text{ZOH}}(t)$ for the range of $t = -2$ to 2. If your answer is correct, you will receive 2.5 points.

6. [2.5%] Continue from the previous question (NOT the alternative question). Suppose we use the optimal band-limited reconstruction and denote the output by $\hat{x}_{\text{sinc}}(t)$. Plot $\hat{x}_{\text{sinc}}(t)$ for the range of $t = -1$ to 3.

   [Alternative question:] If you do not know the answer to this question, you can assume that the sampling period is $T = 1/3$ and the sampled values are $x[n] = \delta[n] + 2\delta[n-2]$. Plot the optimal band-limited reconstruction output $\hat{x}_{\text{sinc}}(t)$ for the range of $t = -2$ to 2. If your answer is correct, you will receive 2.5 points.

7. [2%] Continue from the previous question (NOT the alternative question). Is $\hat{x}_{\text{sinc}}(t) = x(t)$? Justify your answer by one quick sentence.

   [Alternative question:] If you do not know the answer to this question, please under what condition will a sampling system explain the so-called "aliasing" effect. If your answer is correct, you will receive 1.5 points.
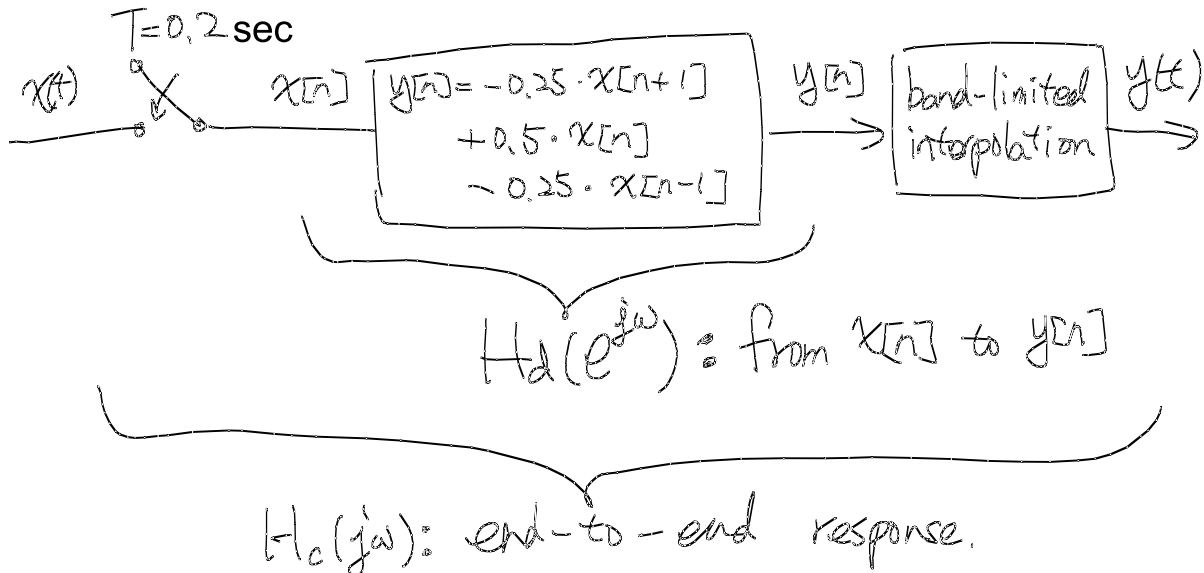
   Consider another signal $y(t) = \frac{\sin(100t) - \sin(20t)}{\pi t}$. Answer the following questions:

8. [2%] What is the value of the "Nyquist frequency" of $y(t)$? Your answer should be something like 44.1 kHz. There is no need to explain how you find the Nyquist frequency. [There is one more question in the back of this page.]

9. [1%] Please choose a sampling period $T$ that ensures that the system is not under-sampled. Your answer should be something like $T = 0.5$ sec. There is no need to explain why you choose such a $T$ value.

*Question 3:* [14%, Work-out question]

Consider the following discrete-time processing system for continuous-time signals.

$T = 0.2$ sec

$x(t)$ →[sampling]→ $x[n]$ → $\boxed{y[n] = -0.25 \cdot x[n+1] + 0.5 \cdot x[n] - 0.25 \cdot x[n-1]}$ → $y[n]$ → $\boxed{\text{band-limited interpolation}}$ → $y(t)$

$H_d(e^{j\omega})$ : from $x[n]$ to $y[n]$

$H_c(j\omega)$ : end-to-end response.

1. [6%] Find the discrete-time frequency response $H_d(e^{j\omega})$ and plot $H_d(e^{j\omega})$ for the range of $-2\pi \le \omega \le 2\pi$.

2. [8%] Find the continuous-time frequency response $H_c(j\omega)$ plot $H_c(j\omega)$ for the range of $-20\pi < \omega < 20\pi$.

   Hint: If you do not know the answer to this question, you can answer the following questions instead: (i) If the input is $x(t) = 5$, what is the output? (ii) What type of filter is the overall system? A low-pass filter? A high-pass filter? Or a band-pass filter? Use a single sentence to justify your answer. If both your answers are correct, you will still get 4 points.

*Question 4:* [13%, Work-out question]

Consider the following discrete-time signal

$$x[n] = \begin{cases} 2^{-n} & \text{if } n \leq 0 \text{ and } n \text{ is even} \\ 0 & \text{if } n \leq 0 \text{ and } n \text{ is odd} \\ 0 & \text{if } n > 0 \end{cases} \tag{1}$$

1. [1%] Plot $x[n]$ for the range of $n = -2$ to 2.

2. [9%] Find the Z-transform of $x[n]$ and carefully specify and draw its ROC.

3. [3%] Draw a separate figure to describe the pole-zero chart of the Z-transform.

   [Alternative question:] If you do not know the answer to the previous question, you can assume $X(z) = \frac{z^2+z}{1+z^2}$ and plot the corresponding pole-zero chart. You will get 3 points if your answer is correct.

*Question 5:* [13%, Work-out question]
Consider a DT LTI system with impulse response

$$h[n] = \sum_{k=0}^{19} 2^{-(k+1)}\delta[n - k]. \tag{2}$$

1. [13%] Find the output $y[n]$ when the input is $x[n] = \cos(3\pi n) + e^{j0.1\pi n}$.

   Hint 1: If the expression of $h[n]$ is confusing to you, you may want to plot $h[n]$ first so that you understand how $h[n]$ looks like.

   Hint 2: You may need the following formula: $\sum_{n=1}^{N} ar^{n-1} = \frac{a(1-r^N)}{1-r}$ if $r \neq 1$.

*Question 6:* [12%, True / False questions. No need for justification]

1. [1%] [T] [F] Two distinctly different discrete-time signals must have different Z-transform *expressions*.

2. [1%] [T] [F] Two distinctly different discrete-time signals can have the same Z-transform *ROCs*.

3. [1%] [T] [F] For any invertible system, two distinctly different input signals $x_1(t)$ and $x_2(t)$ must have two different outputs $y_1(t)$ and $y_2(t)$.

4. [1.5%] [T] [F] For any signal $x[n]$ that has finite power, we can always find its DTFT $X(e^{j\omega})$

5. [1.5%] [T] [F] For any aperiodic signal $x[n]$, its DTFT is also aperiodic.

6. [1.5%] [T] [F] Multiplying by $t$ in the time domain causes multiplication by $j\omega$ in the frequency domain.

7. [1.5%] [T] [F] Multiplying by $e^{jWt}$ in the time domain places the original signal in another frequency band.

8. [1.5%] [T] [F] Consider a signal $y(t) = 5(\mathcal{U}(t+2) - \mathcal{U}(t-2))$. We then have $\int_{-\infty}^{\infty} Y(j\omega)d\omega = 5$.

9. [1.5%] [T] [F] Consider a linear system and we know that when the input is $x(t) = \cos(t)$ the output is $y(t) = \sin(2t)$. The system must be time-varying.

*Question 7:* [15%, Multiple-choice question] Consider two signals

$$h_1(t) = e^{-t^2+1} \cos(t) \tag{3}$$

and

$$h_2[n] = \sin\left(0.1\pi \cdot \left(\int_{n-1}^{n+1} s\, ds\right)\right) + \cos((n+3.5)\pi) \tag{4}$$

1. [1.25%] Is $h_1(t)$ periodic?

2. [1.25%] Is $h_2[n]$ periodic?

3. [1.25%] Is $h_1(t)$ even or odd or neither?

4. [1.25%] Is $h_2[n]$ even or odd or neither?

5. [1.25%] Is $h_1(t)$ of finite energy?

6. [1.25%] Is $h_2[n]$ of finite energy?

Suppose the above two signals are also the impulse responses of two LTI systems: System 1 and System 2, respectively.

1. [1.25%] Is System 1 memoryless?

2. [1.25%] Is System 2 memoryless?

3. [1.25%] Is System 1 causal?

4. [1.25%] Is System 2 causal?

5. [1.25%] Is System 1 stable?

6. [1.25%] Is System 2 stable?

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(j\omega) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

**TABLE 3.1   PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES**

| Property | Section | Periodic Signal | Fourier Series Coefficients |
|---|---|---|---|
| | | $x(t)$ $\left.\right\}$ Periodic with period T and $y(t)$ $\left.\right\}$ fundamental frequency $\omega_0 = 2\pi/T$ | $a_k$ $b_k$ |
| Linearity | 3.5.1 | $Ax(t) + By(t)$ | $Aa_k + Bb_k$ |
| Time Shifting | 3.5.2 | $x(t - t_0)$ | $a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$ |
| Frequency Shifting | | $e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$ | $a_{k-M}$ |
| Conjugation | 3.5.6 | $x^*(t)$ | $a^*_{-k}$ |
| Time Reversal | 3.5.3 | $x(-t)$ | $a_{-k}$ |
| Time Scaling | 3.5.4 | $x(\alpha t), \alpha > 0$ (periodic with period $T/\alpha$) | $a_k$ |
| Periodic Convolution | | $\displaystyle\int_T x(\tau)y(t - \tau)d\tau$ | $Ta_k b_k$ |
| Multiplication | 3.5.5 | $x(t)y(t)$ | $\displaystyle\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$ |
| Differentiation | | $\dfrac{dx(t)}{dt}$ | $jk\omega_0 a_k = jk\dfrac{2\pi}{T}a_k$ |
| Integration | | $\displaystyle\int_{-\infty}^t x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$) | $\left(\dfrac{1}{jk\omega_0}\right)a_k = \left(\dfrac{1}{jk(2\pi/T)}\right)a_k$ |
| Conjugate Symmetry for Real Signals | 3.5.6 | $x(t)$ real | $\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | 3.5.6 | $x(t)$ real and even | $a_k$ real and even |
| Real and Odd Signals | 3.5.6 | $x(t)$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | | $\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} \quad [x(t)\text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} \quad [x(t)\text{ real}] \end{cases}$ | $\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

## Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

## 3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

**TABLE 3.2**    PROPERTIES OF DISCRETE-TIME FOURIER SERIES

| Property | Periodic Signal | Fourier Series Coefficients |
|---|---|---|
| | $x[n]$ } Periodic with period $N$ and <br> $y[n]$ } fundamental frequency $\omega_0 = 2\pi/N$ | $a_k$ } Periodic with <br> $b_k$ } period $N$ |
| Linearity | $Ax[n] + By[n]$ | $Aa_k + Bb_k$ |
| Time Shifting | $x[n - n_0]$ | $a_k e^{-jk(2\pi/N)n_0}$ |
| Frequency Shifting | $e^{jM(2\pi/N)n}x[n]$ | $a_{k-M}$ |
| Conjugation | $x^*[n]$ | $a_{-k}^*$ |
| Time Reversal | $x[-n]$ | $a_{-k}$ |
| Time Scaling | $x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ <br> (periodic with period $mN$) | $\dfrac{1}{m}a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$ |
| Periodic Convolution | $\displaystyle\sum_{r=\langle N\rangle} x[r]y[n - r]$ | $Na_k b_k$ |
| Multiplication | $x[n]y[n]$ | $\displaystyle\sum_{l=\langle N\rangle} a_l b_{k-l}$ |
| First Difference | $x[n] - x[n-1]$ | $(1 - e^{-jk(2\pi/N)})a_k$ |
| Running Sum | $\displaystyle\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$ | $\left( \dfrac{1}{(1 - e^{-jk(2\pi/N)})} \right)a_k$ |
| Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} a_k = a_{-k}^* \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals <br> Real and Odd Signals | $x[n]$ real and even <br> $x[n]$ real and odd | $a_k$ real and even <br> $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | $\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [\text{x}[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [\text{x}[n] \text{ real}] \end{cases}$ | $\mathfrak{Re}\{a_k\}$ <br> $j\mathfrak{Im}\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

## 4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

**TABLE 4.1  PROPERTIES OF THE FOURIER TRANSFORM**

| Section | Property | Aperiodic signal | Fourier transform |
|---|---|---|---|
| | | $x(t)$ | $X(j\omega)$ |
| | | $y(t)$ | $Y(j\omega)$ |
| 4.3.1 | Linearity | $ax(t) + by(t)$ | $aX(j\omega) + bY(j\omega)$ |
| 4.3.2 | Time Shifting | $x(t - t_0)$ | $e^{-j\omega t_0}X(j\omega)$ |
| 4.3.6 | Frequency Shifting | $e^{j\omega_0 t}x(t)$ | $X(j(\omega - \omega_0))$ |
| 4.3.3 | Conjugation | $x^*(t)$ | $X^*(-j\omega)$ |
| 4.3.5 | Time Reversal | $x(-t)$ | $X(-j\omega)$ |
| 4.3.5 | Time and Frequency Scaling | $x(at)$ | $\dfrac{1}{\|a\|}X\left(\dfrac{j\omega}{a}\right)$ |
| 4.4 | Convolution | $x(t) * y(t)$ | $X(j\omega)Y(j\omega)$ |
| 4.5 | Multiplication | $x(t)y(t)$ | $\dfrac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$ |
| 4.3.4 | Differentiation in Time | $\dfrac{d}{dt}x(t)$ | $j\omega X(j\omega)$ |
| 4.3.4 | Integration | $\int_{-\infty}^{t}x(t)dt$ | $\dfrac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$ |
| 4.3.6 | Differentiation in Frequency | $tx(t)$ | $j\dfrac{d}{d\omega}X(j\omega)$ |
| 4.3.3 | Conjugate Symmetry for Real Signals | $x(t)$ real | $\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$ |
| 4.3.3 | Symmetry for Real and Even Signals | $x(t)$ real and even | $X(j\omega)$ real and even |
| 4.3.3 | Symmetry for Real and Odd Signals | $x(t)$ real and odd | $X(j\omega)$ purely imaginary and odd |
| 4.3.3 | Even-Odd Decomposition for Real Signals | $x_e(t) = \mathcal{E}v\{x(t)\}$  [$x(t)$ real]<br>$x_o(t) = \mathcal{O}d\{x(t)\}$  [$x(t)$ real] | $\mathcal{Re}\{X(j\omega)\}$<br>$j\mathcal{Im}\{X(j\omega)\}$ |

4.3.7  Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2 d\omega$$

**TABLE 4.2    BASIC FOURIER TRANSFORM PAIRS**

| Signal | Fourier transform | Fourier series coefficients (if periodic) |
|---|---|---|
| $\displaystyle\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ | $\displaystyle 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$ | $a_k$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ | $a_1 = 1$<br>$a_k = 0,\quad$ otherwise |
| $\cos\omega_0 t$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | $a_1 = a_{-1} = \frac{1}{2}$<br>$a_k = 0,\quad$ otherwise |
| $\sin\omega_0 t$ | $\dfrac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | $a_1 = -a_{-1} = \frac{1}{2j}$<br>$a_k = 0,\quad$ otherwise |
| $x(t) = 1$ | $2\pi\,\delta(\omega)$ | $a_0 = 1,\quad a_k = 0,\ k \neq 0$<br>$\left(\begin{array}{l}\text{this is the Fourier series representation for}\\ \text{any choice of } T > 0\end{array}\right)$ |
| Periodic square wave<br>$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$<br>and<br>$x(t + T) = x(t)$ | $\displaystyle\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\,\delta(\omega - k\omega_0)$ | $\dfrac{\omega_0 T_1}{\pi}\,\mathrm{sinc}\left(\dfrac{k\omega_0 T_1}{\pi}\right) = \dfrac{\sin k\omega_0 T_1}{k\pi}$ |
| $\displaystyle\sum_{n=-\infty}^{+\infty} \delta(t - nT)$ | $\dfrac{2\pi}{T}\displaystyle\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \dfrac{2\pi k}{T}\right)$ | $a_k = \dfrac{1}{T}$ for all $k$ |
| $x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$ | $\dfrac{2\sin\omega T_1}{\omega}$ | — |
| $\dfrac{\sin Wt}{\pi t}$ | $X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$ | — |
| $\delta(t)$ | $1$ | — |
| $u(t)$ | $\dfrac{1}{j\omega} + \pi\delta(\omega)$ | — |
| $\delta(t - t_0)$ | $e^{-j\omega t_0}$ | — |
| $e^{-at}u(t),\ \mathcal{R}e\{a\} > 0$ | $\dfrac{1}{a + j\omega}$ | — |
| $te^{-at}u(t),\ \mathcal{R}e\{a\} > 0$ | $\dfrac{1}{(a + j\omega)^2}$ | — |
| $\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t),$<br>$\mathcal{R}e\{a\} > 0$ | $\dfrac{1}{(a + j\omega)^n}$ | — |

**TABLE 5.1    PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM**

| Section | Property | Aperiodic Signal | Fourier Transform |
|---|---|---|---|
| | | $x[n]$ | $X(e^{j\omega})$ periodic with |
| | | $y[n]$ | $Y(e^{j\omega})$ period $2\pi$ |
| 5.3.2 | Linearity | $ax[n] + by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ |
| 5.3.3 | Time Shifting | $x[n - n_0]$ | $e^{-j\omega n_0}X(e^{j\omega})$ |
| 5.3.3 | Frequency Shifting | $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 5.3.4 | Conjugation | $x^*[n]$ | $X^*(e^{-j\omega})$ |
| 5.3.6 | Time Reversal | $x[-n]$ | $X(e^{-j\omega})$ |
| 5.3.7 | Time Expansion | $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$ | $X(e^{jk\omega})$ |
| 5.4 | Convolution | $x[n] * y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 5.5 | Multiplication | $x[n]y[n]$ | $\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |
| 5.3.5 | Differencing in Time | $x[n] - x[n - 1]$ | $(1 - e^{-j\omega})X(e^{j\omega})$ |
| 5.3.5 | Accumulation | $\displaystyle\sum_{k=-\infty}^{n} x[k]$ | $\dfrac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\displaystyle\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$ |
| 5.3.8 | Differentiation in Frequency | $nx[n]$ | $j\dfrac{dX(e^{j\omega})}{d\omega}$ |
| 5.3.4 | Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ |X(e^{j\omega})| = |X(e^{-j\omega})| \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$ |
| 5.3.4 | Symmetry for Real, Even Signals | $x[n]$ real an even | $X(e^{j\omega})$ real and even |
| 5.3.4 | Symmetry for Real, Odd Signals | $x[n]$ real and odd | $X(e^{j\omega})$ purely imaginary and odd |
| 5.3.4 | Even-odd Decomposition of Real Signals | $x_e[n] = \mathcal{E}v\{x[n]\}$    $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$    $[x[n]$ real$]$ | $\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$ |
| 5.3.9 | Parseval's Relation for Aperiodic Signals $\displaystyle\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \dfrac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$ | | |

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

## 5.7.1  Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients $a_k$ of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence $a_k$ in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence $a_k$ are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

## TABLE 5.2  BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $\displaystyle\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ | $\displaystyle 2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$ | $a_k$ |
| $e^{j\omega_0 n}$ | $\displaystyle 2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi l)$ | (a) $\omega_0 = \frac{2\pi m}{N}$ $\quad a_k = \begin{cases} 1, & k=m,\,m\pm N,\,m\pm 2N,\dots \\ 0, & \text{otherwise} \end{cases}$ <br> (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\cos\omega_0 n$ | $\displaystyle \pi\sum_{l=-\infty}^{+\infty}\{\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi m}{N}$ $\quad a_k = \begin{cases} \frac{1}{2}, & k=\pm m,\,\pm m\pm N,\,\pm m\pm 2N,\dots \\ 0, & \text{otherwise} \end{cases}$ <br> (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\sin\omega_0 n$ | $\displaystyle \frac{\pi}{j}\sum_{l=-\infty}^{+\infty}\{\delta(\omega-\omega_0-2\pi l)-\delta(\omega+\omega_0-2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi r}{N}$ $\quad a_k = \begin{cases} \frac{1}{2j}, & k=r,\,r\pm N,\,r\pm 2N,\dots \\ -\frac{1}{2j}, & k=-r,\,-r\pm N,\,-r\pm 2N,\dots \\ 0, & \text{otherwise} \end{cases}$ <br> (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $x[n]=1$ | $\displaystyle 2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-2\pi l)$ | $a_k = \begin{cases} 1, & k=0,\,\pm N,\,\pm 2N,\dots \\ 0, & \text{otherwise} \end{cases}$ |
| Periodic square wave $x[n]=\begin{cases} 1, & |n|\le N_1 \\ 0, & N_1<|n|\le N/2 \end{cases}$ and $x[n+N]=x[n]$ | $\displaystyle 2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$ | $a_k = \dfrac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N\sin[2\pi k/2N]},\ k\ne 0,\pm N,\pm 2N,\dots$ <br> $a_k = \dfrac{2N_1+1}{N},\ k=0,\pm N,\pm 2N,\dots$ |
| $\displaystyle\sum_{k=-\infty}^{+\infty}\delta[n-kN]$ | $\displaystyle \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$ | $a_k = \dfrac{1}{N}$ for all $k$ |
| $a^n u[n], \quad |a|<1$ | $\dfrac{1}{1-ae^{-j\omega}}$ | — |
| $x[n]=\begin{cases} 1, & |n|\le N_1 \\ 0, & |n|>N_1 \end{cases}$ | $\dfrac{\sin[\omega(N_1+\frac{1}{2})]}{\sin(\omega/2)}$ | — |
| $\dfrac{\sin Wn}{\pi n}=\dfrac{W}{\pi}\,\mathrm{sinc}\left(\dfrac{Wn}{\pi}\right)$ $0<W<\pi$ | $X(\omega)=\begin{cases} 1, & 0\le|\omega|\le W \\ 0, & W<|\omega|\le \pi \end{cases}$ $X(\omega)$ periodic with period $2\pi$ | — |
| $\delta[n]$ | $1$ | — |
| $u[n]$ | $\dfrac{1}{1-e^{-j\omega}}+\displaystyle\sum_{k=-\infty}^{+\infty}\pi\delta(\omega-2\pi k)$ | — |
| $\delta[n-n_0]$ | $e^{-j\omega n_0}$ | — |
| $(n+1)a^n u[n], \quad |a|<1$ | $\dfrac{1}{(1-ae^{-j\omega})^2}$ | — |
| $\dfrac{(n+r-1)!}{n!(r-1)!}a^n u[n], \quad |a|<1$ | $\dfrac{1}{(1-ae^{-j\omega})^r}$ | — |

**TABLE 10.1** PROPERTIES OF THE z-TRANSFORM

| Section | Property | Signal | z-Transform | ROC |
|---|---|---|---|---|
| | | $x[n]$ | $X(z)$ | $R$ |
| | | $x_1[n]$ | $X_1(z)$ | $R_1$ |
| | | $x_2[n]$ | $X_2(z)$ | $R_2$ |
| 10.5.1 | Linearity | $ax_1[n] + bx_2[n]$ | $aX_1(z) + bX_2(z)$ | At least the intersection of $R_1$ and $R_2$ |
| 10.5.2 | Time shifting | $x[n - n_0]$ | $z^{-n_0}X(z)$ | $R$, except for the possible addition or deletion of the origin |
| 10.5.3 | Scaling in the z-domain | $e^{j\omega_0 n}x[n]$ | $X(e^{-j\omega_0}z)$ | $R$ |
| | | $z_0^n x[n]$ | $X\left(\dfrac{z}{z_0}\right)$ | $z_0 R$ |
| | | $a^n x[n]$ | $X(a^{-1}z)$ | Scaled version of $R$ (i.e., $|a|R =$ the set of points $\{|a|z\}$ for z in $R$) |
| 10.5.4 | Time reversal | $x[-n]$ | $X(z^{-1})$ | Inverted $R$ (i.e., $R^{-1} =$ the set of points $z^{-1}$, where z is in $R$) |
| 10.5.5 | Time expansion | $x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer $r$ | $X(z^k)$ | $R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in $R$) |
| 10.5.6 | Conjugation | $x^*[n]$ | $X^*(z^*)$ | $R$ |
| 10.5.7 | Convolution | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ | At least the intersection of $R_1$ and $R_2$ |
| 10.5.7 | First difference | $x[n] - x[n - 1]$ | $(1 - z^{-1})X(z)$ | At least the intersection of $R$ and $|z| > 0$ |
| 10.5.7 | Accumulation | $\sum_{k=-\infty}^{n} x[k]$ | $\dfrac{1}{1 - z^{-1}}X(z)$ | At least the intersection of $R$ and $|z| > 1$ |
| 10.5.8 | Differentiation in the z-domain | $nx[n]$ | $-z\dfrac{dX(z)}{dz}$ | $R$ |
| 10.5.9 | | Initial Value Theorem<br>If $x[n] = 0$ for $n < 0$, then<br>$x[0] = \lim_{z \to \infty} X(z)$ | | |

**TABLE 10.2**    SOME COMMON *z*-TRANSFORM PAIRS

| Signal | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\dfrac{1}{1 - z^{-1}}$ | $\lvert z \rvert > 1$ |
| 3. $-u[-n - 1]$ | $\dfrac{1}{1 - z^{-1}}$ | $\lvert z \rvert < 1$ |
| 4. $\delta[n - m]$ | $z^{-m}$ | All $z$, except $0$ (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $\alpha^n u[n]$ | $\dfrac{1}{1 - \alpha z^{-1}}$ | $\lvert z \rvert > \lvert \alpha \rvert$ |
| 6. $-\alpha^n u[-n - 1]$ | $\dfrac{1}{1 - \alpha z^{-1}}$ | $\lvert z \rvert < \lvert \alpha \rvert$ |
| 7. $n\alpha^n u[n]$ | $\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$ | $\lvert z \rvert > \lvert \alpha \rvert$ |
| 8. $-n\alpha^n u[-n - 1]$ | $\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$ | $\lvert z \rvert < \lvert \alpha \rvert$ |
| 9. $[\cos \omega_0 n] u[n]$ | $\dfrac{1 - [\cos \omega_0] z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + z^{-2}}$ | $\lvert z \rvert > 1$ |
| 10. $[\sin \omega_0 n] u[n]$ | $\dfrac{[\sin \omega_0] z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + z^{-2}}$ | $\lvert z \rvert > 1$ |
| 11. $[r^n \cos \omega_0 n] u[n]$ | $\dfrac{1 - [r \cos \omega_0] z^{-1}}{1 - [2r \cos \omega_0] z^{-1} + r^2 z^{-2}}$ | $\lvert z \rvert > r$ |
| 12. $[r^n \sin \omega_0 n] u[n]$ | $\dfrac{[r \sin \omega_0] z^{-1}}{1 - [2r \cos \omega_0] z^{-1} + r^2 z^{-2}}$ | $\lvert z \rvert > r$ |

## 10.7.1 Causality

A causal LTI system has an impulse response $h[n]$ that is zero for $n < 0$, and therefore is right-sided. From Property 4 in Section 10.2 we then know that the ROC of $H(z)$ is the exterior of a circle in the $z$-plane. For some systems, e.g., if $h[n] = \delta[n]$, so that $H(z) = 1$, the ROC can extend all the way in to and possibly include the origin. Also, in general, for a right-sided impulse response, the ROC may or may not include infinity. For example, if $h[n] = \delta[n + 1]$, then $H(z) = z$, which has a pole at infinity. However, as we saw in Property 8 in Section 10.2, for a causal system the power series

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

does not include any positive powers of $z$. Consequently, the ROC includes infinity. Summarizing, we have the follow principle:

> A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.