Midterm #3 of ECE301, Prof. Wang's section

6:30–7:30pm Tuesday, April 05, 2012, FRNY G140,

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solution

Student ID:

E-mail:

Signature:

Question 1: [20%, Work-out question, Outcome 4] Consider a discrete-time periodic signal $x[n]$, which is periodic with period 40 and

$$x[n] = \begin{cases} 1 & \text{if } 1 \leq n \leq 10 \\ 0 & \text{if } 11 \leq n \leq 30 \\ 2 & \text{if } 31 \leq n \leq 40 \end{cases}. \tag{1}$$

We use $a_0, a_1, \cdots, a_{39}$ to denote the discrete-time Fourier series coefficients of $x[n]$

1. [4%] What is the value of a_0?
2. [4%] What is the value of a_{20}?
3. [4%] What is the value of $a_0 + a_1 + a_2 + a_3 + \cdots + a_{39}$?
4. [4%] What is the value of $(a_0)^2 + (a_1)^2 + (a_2)^2 + \cdots + (a_{39})^2$?
5. [4%] What is the value of $a_0 - a_1 + a_2 - a_3 + \cdots - a_{39}$?

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{40} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{\pi}{20}n} = \frac{1}{40} \sum_{n=1}^{10} \left(e^{-jk\frac{\pi}{20}}\right)^n + \frac{2}{40} \sum_{n=31}^{40} \left(e^{-jk\frac{\pi}{20}}\right)^n$$

$$= \frac{e^{-jk\frac{\pi}{20}}}{40} \frac{1 - e^{-jk\frac{\pi}{20}\cdot 10}}{1 - e^{-jk\frac{\pi}{20}}} + \frac{2}{40} e^{-jk\frac{\pi}{20}\cdot 31} \frac{1 - e^{-jk\frac{\pi}{20}\cdot 10}}{1 - e^{-jk\frac{\pi}{20}}}$$

$$= \left(\frac{1}{40} e^{-jk\frac{\pi}{20}} + \frac{1}{20} e^{-jk\frac{31}{20}\pi} \right) \frac{1 - e^{-jk\frac{\pi}{2}}}{1 - e^{-jk\frac{\pi}{20}}} \quad \text{when } k \neq 0$$

When $k = 0$, $a_0 = \frac{1}{40} \sum_{n=\langle N \rangle} x[n] = \frac{1 \cdot 10 + 2 \cdot 10}{40} = \frac{3}{4}$.

1. $a_0 = \frac{3}{4}$.

2. $a_{20} = \left(\frac{1}{40} e^{-j\pi} + \frac{1}{20} e^{-j31\pi} \right) \frac{1 - e^{-j10\pi}}{1 - e^{-j\pi}} = \left(-\frac{1}{40} - \frac{1}{20} \right) \frac{0}{2} = 0$

3. Note that $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}$. Thus, $x[n] = \sum_{k=0}^{39} a_k e^{jk\frac{2\pi}{40}n}$

$x[0] = \sum_{k=0}^{39} a_k$, Hence, $a_0 + a_1 + \cdots + a_{39} = x[0] = 2$.

4. Using Parseval's Relation, $\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2 = \sum_{k=\langle N\rangle}|a_k|^2$.

$\therefore\ (a_0)^2 + (a_1)^2 + \cdots + (a_{39})^2 = \frac{1}{40}\sum_{n=0}^{39}|x[n]|^2 = \frac{1}{40}\left(10\cdot 1 + 10\cdot 4\right) = \frac{5}{4}$

5. From $x[n] = \sum_{k=0}^{39} a_k e^{jk\frac{2\pi}{40}n}$,

$x[20] = \sum_{k=0}^{39} a_k e^{jk\pi} = \sum_{k=0}^{39} a_k (e^{j\pi})^k = \sum_{k=0}^{39} a_k (-1)^k$

$\therefore\ a_0 - a_1 + a_2 - \cdots + a_{38} - a_{39} = x[20] = 0$.

Question 2: [10%, Work-out question, Outcomes 2, 4, and 5] The input and the output of a stable and causal LTI system are described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 10y(t) = 2x(t). \tag{2}$$

1. [10%] Find the impulse response $h(t)$ of this system.

Following linearity, change into CTFT: $(j\omega)^2 Y(j\omega) + 7(j\omega)Y(j\omega) + 10Y(j\omega) = 2X(j\omega)$

$$\therefore H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{(j\omega)^2 + 7(j\omega) + 10} = \frac{2}{(5+j\omega)(2+j\omega)}$$

$$= \frac{A}{5+j\omega} + \frac{B}{2+j\omega} \Rightarrow \begin{array}{l} A = -\frac{2}{3}. \\ B = \frac{2}{3} \end{array}$$

$$\therefore H(j\omega) = -\frac{2}{3}\cdot\left(\frac{1}{5+j\omega}\right) + \frac{2}{3}\left(\frac{1}{2+j\omega}\right)$$

$\Updownarrow$ CTFT

$$\boxed{h(t) = -\frac{2}{3}e^{-5t}u(t) + \frac{2}{3}e^{-2t}u(t).}$$

Note that LTI system is causal & stable

since $h(t) = 0 \ \forall t < 0$, $\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$

Question 3: [20%, Work-out question, Outcomes 3, 4, and 5] Consider an LTI system with impulse response $h(t) = \frac{\sin(3t)\sin(6t)}{t^2}$. Answer the following questions.

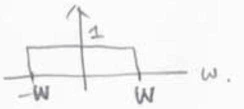
1. [10%] Plot the frequency response $H(j\omega)$ for the range of $-10 \le \omega \le 10$?
2. [10%] Find out the output $y(t)$ when the input is $x(t) = \cos(9t) + \sin(6t+\pi) + e^{j3t} + 2$.

   Hint: If you do not know the answer to the previous subquestion, you can assume that

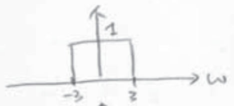
$$H(j\omega) = \begin{cases} \omega + 9 & \text{if } -9 < \omega < 0 \\ 9 - \omega & \text{if } 0 < \omega < 9 \\ 0 & \text{otherwise} \end{cases} . \tag{3}$$

   You will get 9 points if your answer is correct.

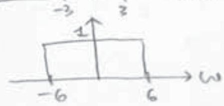
1. From table, $\frac{\sin Wt}{\pi t} \xleftrightarrow{CTFT}$



Let $h_1(t) = \frac{\sin(3t)}{\pi t} \xleftrightarrow{CTFT} H_1(j\omega) =$



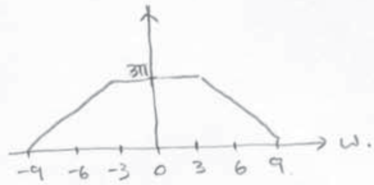
$h_2(t) = \frac{\sin(6t)}{\pi t} \xleftrightarrow{CTFT} H_2(j\omega) =$



Then, $h(t) = \pi^2 \cdot h_1(t) \cdot h_2(t)$

$\updownarrow$ CTFT Multiplication property.

$H(j\omega) = \pi^2 \cdot \frac{1}{2\pi} \cdot H_1(j\omega) \circledast H_2(j\omega) =$



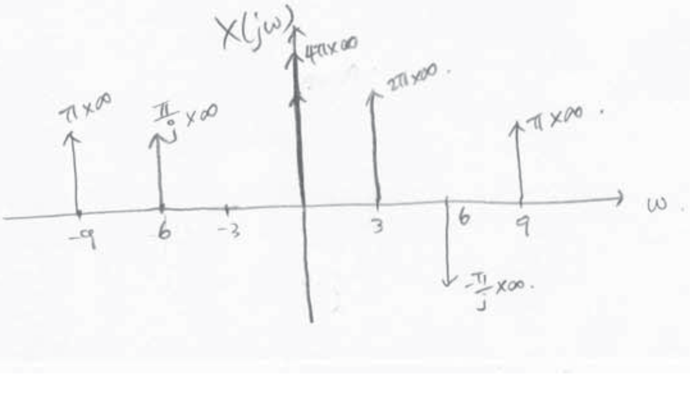
2. $x(t) = \cos(9t) + \sin(6t+\pi) + e^{j3t} + 2$

$= \frac{1}{2}e^{j9t} + \frac{1}{2}e^{-j9t} + \frac{1}{2j}e^{j\pi}e^{j6t} - \frac{1}{2j}e^{-j\pi}e^{-j6t} + e^{j3t} + 2.$

CTFT $\updownarrow$

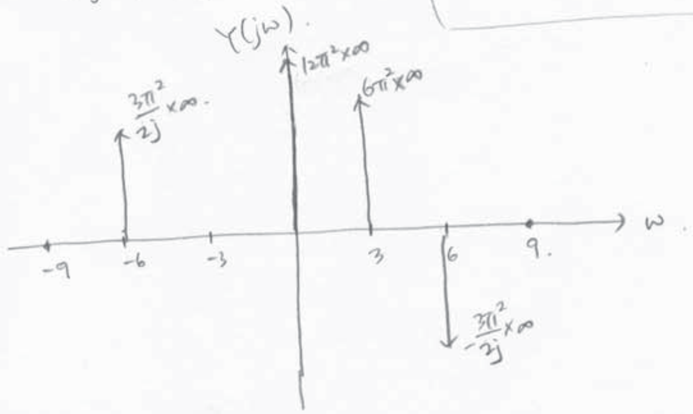
$X(j\omega) = \pi\delta(\omega-9) + \pi\delta(\omega+9) + \frac{\pi}{j}e^{j\pi}\delta(\omega-6) - \frac{\pi}{j}e^{-j\pi}\delta(\omega+6)$

$+ 2\pi\delta(\omega-3) + 4\pi\delta(\omega)$.

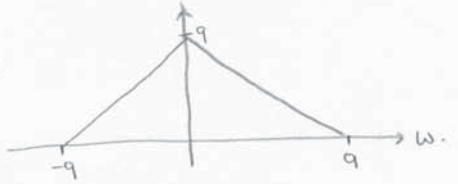


$$Y(j\omega) = X(j\omega)\,H(j\omega)$$

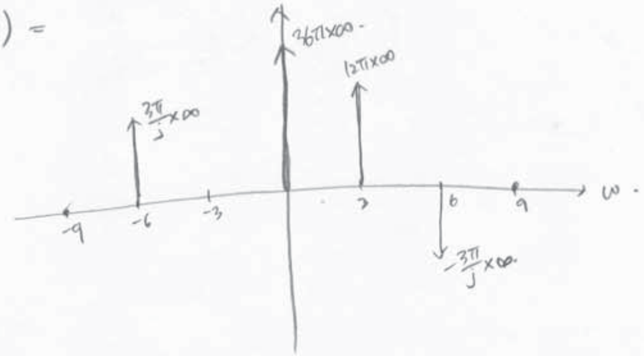
$$\boxed{\therefore\ y(t) = 6\pi + 3\pi e^{j3t} - \frac{3\pi}{2}\sin(6t)}$$



[Alterative] $H(j\omega) =$



$\therefore\ Y(j\omega) =$



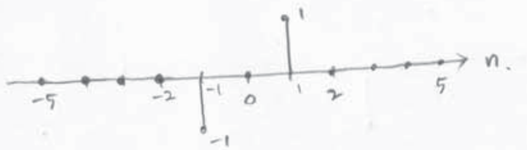
$$\boxed{\therefore\ y(t) = 18 + 6e^{j3t} - 3\sin(6t)}$$

Question 4: [20%, Work-out question, Outcomes 4 and 6] Consider a discrete-time ~~period~~ signal $x[n]$ as follows

$$x[n] = \begin{cases} \sin\left(\frac{\pi n}{2}\right) & \text{if } -2 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

1. [5%] Plot $x[n]$ for the range of $-5 \le n \le 5$.
2. [9%] Compute its discrete-time Fourier transform (DTFT) $X(e^{j\omega})$.
3. [6%] Is $X(e^{j\omega})$ periodic? Plot $X(e^{j\omega})$ for the range of $-2\pi \le \omega \le 2\pi$.

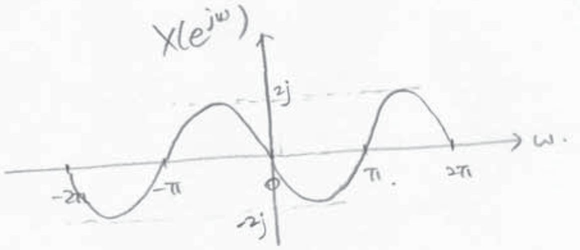
1.



2. $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = e^{-j\omega} - e^{+j\omega} = -2j\left(\frac{e^{j\omega}}{2j} - \frac{e^{-j\omega}}{2j}\right)$

$= -2j \sin(\omega)$

3. Yes.



Question 5: [30%, Work-out question, Outcomes 3, 4, and 5] Consider an AM system, which sends the input signal $x(t)$ over a cosine carrier of frequency 1 Hz.

More specifically, we denote the input signal as $x(t)$ and use $y(t)$ to denote the AM modulated signal, which will be sent out by the AM transmitter.

1. [5%] What is the value of the carrier frequency ω_c with the unit being (rad/sec)? Write down the input/output relationship (equation) between $x(t)$ and $y(t)$.

2. [5%] The receiver uses *synchronous demodulation*. Let $w(t)$ denote the resulting signal after demodulation. Write down the relationship between $y(t)$ and $w(t)$. Your answer should consist of statements like "multiplying ....." and/or "using a filter....." Please be specific about the parameters of the filters. If you prefer, you can also use a block diagram (flow chart) to describe your demodulation system instead of using sentences.

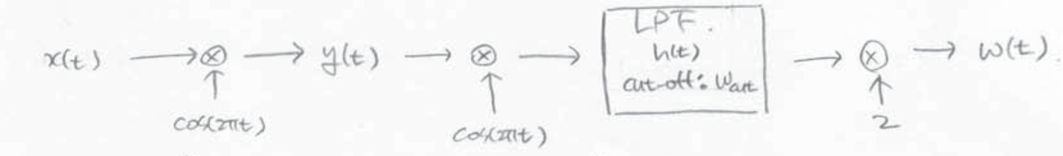
3. [10%] Suppose we also know that $x(t) = \frac{\sin(3\pi t)}{\pi t}$, plot the Fourier transforms $X(j\omega)$ and $Y(j\omega)$.

4. [5%] Answer the question: "Can the receiver *demodulate* the original signal $x(t)$ from the received signal $y(t)$?" Write down your reasons in one or two short sentences.

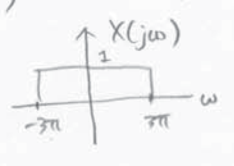
5. [5%] Suppose that we are allowed to increase the carrier frequency to f_{carrier} Hz. What is the minimum value of f_{carrier} for which the receiver can successfully demodulate $x(t)$ (as defined in sub-question 3) from $y(t)$?

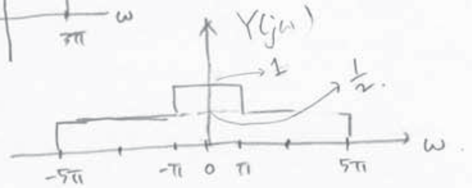
$f_c = 1\,\text{Hz}$

1. $\omega_c = 2\pi \cdot f_c = 2\pi$ (rad/sec) $y(t) = x(t)\cos(2\pi t)$.

2. 

$w(t) = \left\{\left(y(t)\cdot\cos(2\pi t)\right) \circledast h(t)\right\} \times 2$

3. $x(t) = \frac{\sin(3\pi t)}{\pi t} \xleftrightarrow{\text{CTFT}}$ 

$y(t) = \cos(2\pi t)\,x(t) \xleftrightarrow{\text{CTFT}}$ 

4. No. $y(t)$ contains the distorted $x(t)$.

5. We need to modulate $x(t)$ with at least 3π (rad/sec) due to BW of $x(t)$.

$\therefore$ minimum value of $f_c = \frac{3}{2}$ Hz.