Midterm #2 of ECE301, Prof. Wang's section

6:30–7:30pm Tuesday, February 28, 2012, FRNY G140,

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 10 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solution

Student ID:

E-mail:

Signature:

Question 1: [35%, Work-out question, Outcomes 2 and 3] Consider a discrete-time linear time-invariant system with the impulse response being

$$h[n] = 2^{-|n|}. \tag{1}$$

Find out the output $y[n]$ when the input is

1. [3%] Find out the output $y[n]$ when the input is

$$x[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

2. [20%] Find out the output $y[n]$ when the input is

$$x[n] = \begin{cases} 2^n & \text{if } -10 \le n \le 10 \\ 0 & \text{otherwise} \end{cases}. \tag{3}$$

3. [12%] Find out the output $y[n]$ when the input is $x[n] = e^{j(n+2)}$.

Hint: The geometric series formulas are

$$\text{if } |r| < 1, \text{ then } \sum_{k=1}^{\infty} a_0 r^{k-1} = \frac{a_0}{1-r} \tag{4}$$

$$\text{if } r \neq 1, \text{ then } \sum_{k=1}^{K} a_0 r^{k-1} = \frac{a_0(1-r^K)}{1-r}. \tag{5}$$

1. $x[n] = \delta[n] \xrightarrow[h[n]]{LTI} y[n] = h[n] = 2^{-|n|}$

2. $y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k].$

$= \sum_{k=-10}^{10} 2^k\, 2^{-|n-k|} = \sum_{k=-10}^{10} 2^k\, 2^{-|k-n|}$

$$= \begin{cases} \sum_{k=-10}^{10} 2^k \cdot 2^{-(k-n)}. & n < -10. \\ \sum_{k=-10}^{n} 2^k \cdot 2^{+(k-n)} + \sum_{k=n+1}^{10} 2^k \cdot 2^{-(k-n)} & -10 \le n < 10 \\ \sum_{k=-10}^{10} 2^k \cdot 2^{+(k-n)} & n \ge 10. \end{cases}$$

$$= \begin{cases} 21 \cdot 2^n & n < -10. \\ 2^{-n} \cdot \underbrace{\sum_{k=-10}^{n} 4^k}_{\substack{\shortparallel \\ 4^{-10} \cdot \sum_{k=1}^{n+11} 4^{k-1} = 4^{-10} \cdot \frac{1-4^{n+11}}{1-4} = \frac{1}{3}\left(4^{n+1} - 4^{-10}\right)}} + (10-n) \cdot 2^n & -10 \le n < 10. \\ 2^{-n} \underbrace{\sum_{k=-10}^{10} 4^k}_{\substack{\shortparallel \\ 4^{-10} \sum_{k=1}^{21} 4^{k-1} = 4^{-10} \cdot \frac{1-4^{21}}{1-4} = \frac{1}{3}\left(4^{11} - 4^{-10}\right)}} & n \ge 10. \end{cases}$$

$$= \begin{cases} 21 \cdot 2^n & n < -10. \\ \frac{1}{3}\left(4^{n+1} - 4^{-10}\right) \cdot 2^{-n} + (10-n) \cdot 2^n & -10 \le n < 10. \\ \frac{1}{3}\left(4^{11} - 4^{-10}\right) \cdot 2^{-n} & n \ge 10. \end{cases}$$

3. $x[n] = e^{j(n+2)} = e^{jn} \cdot e^{j2}.$

※ $x[n] = e^{j\omega n} \xrightarrow[h[n]]{LTI} y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

$$= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}\, h[k] = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} e^{-j\omega k} h[k]}_{H(e^{j\omega})}$$

Thus, find $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} e^{-j\omega k}\, h[k]$

$$= \sum_{k=-\infty}^{\infty} e^{-j\omega k}\, 2^{-|k|}$$

$$= \sum_{k=0}^{\infty} e^{-j\omega k}\, 2^{-k} + \sum_{k=-\infty}^{-1} e^{-j\omega k}\, 2^{+k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2e^{j\omega}}\right)^k + \sum_{k=1}^{\infty} \left(\frac{e^{j\omega}}{2}\right)^k$$

Note that $\left|\frac{1}{2e^{j\omega}}\right| < 1 \quad \left|\frac{e^{j\omega}}{2}\right| < 1$

$$= \frac{1}{1 - \frac{1}{2}e^{j\omega}} + \left(\frac{e^{j\omega}}{2}\right) \cdot \frac{1}{1 - e^{j\omega}/2}$$

$$= \frac{2e^{j\omega}}{2e^{j\omega} - 1} + \frac{e^{j\omega}}{2 - e^{j\omega}}$$

By LTI, when $x[n] = e^{j2} \cdot e^{jn}$

then $y[n] = e^{j2} \cdot H(e^{j\omega})\Big|_{\omega=1} \cdot e^{jn}$

$$= e^{j2} \cdot \left(\frac{2e^{j}}{2e^{j} - 1} + \frac{e^{j}}{2 - e^{j}} \right) \cdot e^{jn}$$

Question 2: [15%, Work-out question, Outcome 4] $y[n] = \sin\left(\frac{4\pi n}{5}\right) - e^{j\frac{2\pi n}{3}}$. Find the Fourier series of $y[n]$.

Fundamental Period of $\sin\left(\frac{4\pi n}{5}\right) = 5.$

" of $e^{j\frac{2\pi n}{3}} = 3.$

LCM $\Rightarrow$ $y[n]$ is periodic with 15.

$$\Rightarrow \quad y[n] = \sum_{k=\langle 15 \rangle} a_k e^{jk\frac{2\pi}{15}n}$$

But using Euler's,

$$\sin\left(\frac{4\pi n}{5}\right) = \frac{1}{2j} e^{j\frac{4\pi n}{5}} - \frac{1}{2j} e^{-j\frac{4\pi}{5}n}.$$

$$= \frac{1}{2j} e^{j\,6\cdot\frac{2\pi}{15}n} - \frac{1}{2j} e^{j(-6)\cdot\frac{2\pi}{15}n}$$

$$e^{j\frac{2\pi}{3}n} = e^{j(5)\cdot\frac{2\pi}{15}n}.$$

Thus,

$$a_k = \begin{cases} 1/2j & k=6 \\ -1/2j & k=-6 \text{ or } 9 \text{ (depends on the range of } a_k \text{ you choose)} \\ -1 & k=5 \\ 0 & \text{otherwise.} \end{cases}$$

Question 3: [15%, Work-out question, Outcome 4]

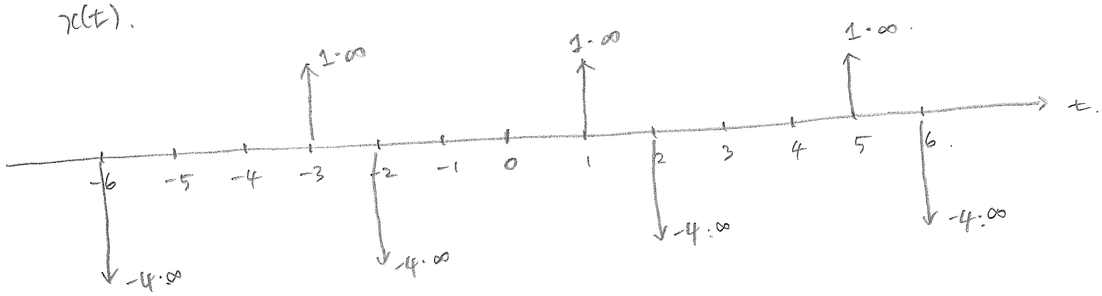
1. [5%] $x(t)$ is periodic with period $T = 4$ and

$$x(t) = \delta(t-1) - t^2 \cdot \delta(t-2) \quad \text{if } 0 \le t \le 4. \tag{6}$$

   Plot $x(t)$ for the range of $t = -6$ to 6.

2. [10%] Compute the Fourier series of $x(t)$.

1.



2. $a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$

$= \frac{1}{4} \int_0^4 \left(\delta(t-1) - t^2 \delta(t-2) \right) e^{-jk\frac{\pi}{2}t}\, dt$

$= \frac{1}{4} \int_0^4 \delta(t-1) e^{-jk\frac{\pi}{2}t}\, dt - \frac{1}{4} \int_0^4 t^2 \delta(t-2)\, e^{-jk\frac{\pi}{2}t}\, dt.$

$= \frac{1}{4} e^{-jk\frac{\pi}{2}} - \frac{1}{4} \cdot 4 e^{-jk\pi}$

$= \frac{1}{4} (-j)^k - (-1)^k$ when $k \neq 0$.

$a_0 = \frac{1}{4} \int_0^4 x(t)\, dt = -\frac{3}{4}$

$\therefore$ $a_k = \frac{1}{4} (-j)^k - (-1)^k$ for all k.

Question 4: [15%, Work-out question, Outcome 4]
Consider a signal $z(t)$, which is periodic with period $T = 4$ and

$$z(t) = \begin{cases} 1 & \text{if } -1 \leq t \leq 1 \\ 0 & \text{if } 1 \leq t < 3 \end{cases}. \tag{7}$$

We know that the Fourier series coefficients of $z(t)$ are

$$b_k = \begin{cases} \frac{1}{2} & \text{if } k = 0 \\ \frac{\sin(\frac{k\pi}{2})}{k\pi} & \text{if } k \neq 0 \end{cases}. \tag{8}$$

Answer the following questions

1. [12%, 2%, 1%] Compute the values of (i) $\sum_{k=-\infty}^{\infty} b_k \cdot e^{-jk\frac{\pi}{3}}$; (ii) $\sum_{k=-\infty}^{\infty} b_k \cdot \cos(k\frac{\pi}{3})$; and (iii) $\sum_{k=-\infty}^{\infty} b_k \cdot \sin(k\frac{\pi}{3})$.

$z(t) \xleftrightarrow{F.S.} b_k$. Let $c_k = b_k \cdot e^{-jk\frac{\pi}{3}} = b_k \cdot e^{-jk\frac{2\pi}{4}\cdot(\frac{2}{3})}$

$d_k = b_k \cdot e^{+jk\frac{\pi}{3}} = b_k \cdot e^{-jk\frac{2\pi}{4}\cdot(-\frac{2}{3})}$

$\therefore\ z(t-\frac{2}{3}) \xleftrightarrow{F.S.} c_k$, $z(t+\frac{2}{3}) \xleftrightarrow{F.S.} d_k$.

(i). From $z(t-\frac{2}{3}) = \sum_{k=-\infty}^{\infty} c_k e^{jk\frac{2\pi}{4}t}$,

at $t=0$, $z(-\frac{2}{3}) = \sum_{k=-\infty}^{\infty} c_k = \sum_{k=-\infty}^{\infty} b_k e^{-jk\frac{\pi}{3}}$

$= 1$.

(ii). Note that $b_k \cdot \cos(k\frac{\pi}{3}) = \frac{1}{2} b_k e^{jk\frac{\pi}{3}} + \frac{1}{2} b_k e^{-jk\frac{\pi}{3}}$

$= \frac{1}{2} d_k + \frac{1}{2} c_k$.

$\therefore\ \frac{1}{2} z(t+\frac{2}{3}) + \frac{1}{2} z(t-\frac{2}{3}) \xleftrightarrow{F.S.} \frac{1}{2} d_k + \frac{1}{2} c_k$.

Similarily to (i), $\sum_{k=-\infty}^{\infty} \left(\frac{1}{2} d_k + \frac{1}{2} c_k\right) = \frac{1}{2} z(\frac{2}{3}) + \frac{1}{2} z(-\frac{2}{3})$

$= 1$.

(iii) $$b_k \cdot \sin\left(k\frac{\pi}{3}\right) = \frac{1}{2j} b_k e^{jk\frac{\pi}{3}} - \frac{1}{2j} b_k e^{-jk\frac{\pi}{3}}$$

$$= \frac{1}{2j} d_k - \frac{1}{2j} c_k.$$

$$\therefore \quad \frac{1}{2j} z\left(t+\frac{2}{3}\right) - \frac{1}{2j} z\left(t-\frac{2}{3}\right) \xleftrightarrow{\text{F.S.}} \frac{1}{2j} d_k - \frac{1}{2j} c_k.$$

$$\therefore \quad \sum_{k=-\infty}^{\infty} \left(\frac{1}{2j} d_k - \frac{1}{2j} c_k \right) = \frac{1}{2j} z\left(+\frac{2}{3}\right) - \frac{1}{2j} z\left(-\frac{2}{3}\right)$$

$$= 0.$$

※ Easier solution to (ii) and (iii).

Note that $$\mathrm{Re}\left\{ \sum_{k=-\infty}^{\infty} b_k e^{-jk\frac{\pi}{3}} \right\} = \sum_{k=-\infty}^{\infty} b_k \cos\left(-k\frac{\pi}{3}\right) = \sum_{k=-\infty}^{\infty} b_k \cos\left(k\frac{\pi}{3}\right).$$

$$\mathrm{Im}\left\{ \sum_{k=-\infty}^{\infty} b_k e^{-jk\frac{\pi}{3}} \right\} = \sum_{k=-\infty}^{\infty} b_k \sin\left(-k\frac{\pi}{3}\right) = -\sum_{k=-\infty}^{\infty} b_k \sin\left(k\frac{\pi}{3}\right).$$

Thus, the solution to (ii) will be $\mathrm{Re}\{1\} = 1$.

" to (iii) will be $-\mathrm{Im}\{1\} = 0$.

Question 5: [20%, Multiple Choices] The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t) = 0$, the output is $y_1(t) = 0$. For general input $x_1(t)$, the output is

$$y_1(t) = y_1(t-1) + x_1(2t). \tag{9}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = x_2[-n^2]. \tag{10}$$

Answer the following questions

1. [4%, Outcome 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Outcome 1] Is System 1 causal? Is System 2 causal?
3. [4%, Outcome 1] Is System 1 stable? Is System 2 stable?
4. [4%, Outcome 1] Is System 1 linear? Is System 2 linear?
5. [4%, Outcome 1] Is System 1 time-invariant? Is System 2 time-invariant?

System 1.	System 2
Memory	Memory.
Non causal.	Causal.
Not Stable.	Stable
Linear.	Linear.
Time-variant.	Time-Variant.