Midterm #3 of ECE301, Prof. Wang's section

8–9pm Wednesday, November 14, 2012, ME 1061,

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

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Signature:

Question 1: [20%, Work-out question, Learning Objectives 3, 4, and 5] Consider the following two signals $x[n]$ and $y[n]$:

$$x[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq 9 \\ 0 & \text{if } 10 \leq n \leq 19 \\ \text{periodic with period 20} & \end{cases}$$

$$y[n] = \sin\left(\frac{3\pi}{10}n\right)$$

and we know that $z[n] = x[n] \cdot y[n]$.

1. [8%] Find the discrete-time Fourier series a_k of $x[n]$.
2. [5%] Find the discrete-time Fourier series b_k of $y[n]$.
3. [7%] Denote the discrete-time Fourier series of $z[n]$ by c_k. Find the value of c_4.

Hint: You may need the following formula: When $r \neq 1$, we have $\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}$.

1. $$a_K = \frac{1}{20} \sum_{n=0}^{9} e^{-j\frac{2\pi}{20}Kn}$$

$$= \frac{1}{20} \sum_{n=0}^{9} \left(e^{-j\frac{\pi}{10}K}\right)^n$$

$$= \frac{1}{20} \frac{1 - e^{-j\frac{\pi}{10}K(10)}}{1 - e^{-j\frac{\pi}{10}K}}$$

$$= \frac{1}{20} \frac{1 - e^{-j\pi K}}{1 - e^{-j\frac{\pi}{10}K}} = \frac{1}{20} \frac{e^{-j\frac{\pi}{2}K}}{e^{-j\frac{\pi}{20}K}} \frac{\sin\left(\frac{\pi}{2}K\right)}{\sin\left(\frac{\pi}{20}K\right)}$$

$$= \frac{1}{20} e^{-j\frac{9}{20}\pi K} \frac{\sin\left(\frac{\pi}{2}K\right)}{\sin\left(\frac{\pi}{20}K\right)}, \quad \text{for} \quad 0 < K \leq 19$$

$$a_0 = \frac{1}{20} \sum_{n=0}^{9} 1 = \frac{1}{20} \times 10 = \frac{1}{2}$$

2. $y[n] = \sin\left(\frac{3\pi}{10} n\right)$

$= \sin\left(\frac{2\pi(3)}{20} n\right)$

$\Rightarrow$ periodic with period $N = 20$

$$y[n] = \frac{1}{2j} e^{+j\frac{2\pi(3)}{20} n} - \frac{1}{2j} e^{-j\frac{2\pi(3)}{20} n}$$

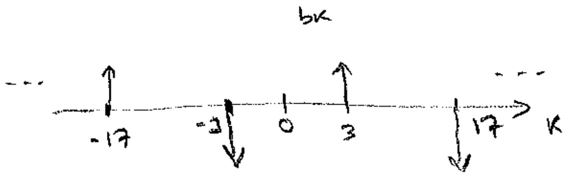
$$= \frac{1}{2j} e^{j\frac{2\pi(3)}{20} n} - \frac{1}{2j} e^{-j\frac{2\pi(3)}{20} n} \cdot e^{j2\pi n}$$

$$= \frac{1}{2j} e^{j\frac{2\pi(3)}{20} n} - \frac{1}{2j}\, {}^{j\frac{2\pi(17)}{20} n}$$

$$\therefore \quad b_k = \begin{cases} \frac{1}{2j}, & \text{for } k = 3 \\ -1/2j, & \text{for } k = 17 \\ 0, & \text{for other } 0 \le k \le 19 \end{cases}$$

3. Since $x[n]$ and $y[n]$ have equal periods, we can apply the multiplication property of the DT FS:

$$c_k = \sum_{\ell=0}^{19} a_\ell b_{k-\ell}$$



$$c_4 = \sum_{\ell=0}^{19} a_\ell b_{4-\ell}$$

$-15 \le 4-\ell \le 4$
$\Rightarrow$ @ $4-\ell = -3 \Leftrightarrow \ell = 7$
@ $4-\ell = 3 \Leftrightarrow \ell = 1$

$$= a_1 b_3 + a_7 b_{-3}$$

$$= \frac{1}{40j} e^{-j\frac{9}{20}\pi} \frac{1}{\sin(\frac{\pi}{20})} + \frac{1}{40j} e^{-j\frac{63}{20}\pi} \frac{1}{\sin\left(\frac{7\pi}{20}\right)}$$

Question 2: [20%, Work-out question, Learning Objective 4] Consider a Fourier transform pair $(x(t), X(j\omega))$. We know that

$$X(j\omega) = \frac{\cos(4\omega)\sin(2\omega)}{\omega}$$

1. [5%] Find the value of $\int_{-\infty}^{\infty} x(t)dt$?
2. [5%] Find the value of $\int_{-\infty}^{\infty} x(t)e^{j\frac{\pi}{2}t}dt$?
3. [10%] Plot $x(t)$ for the range of $-7 \leq t \leq 7$.

1. $$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow \int_{-\infty}^{\infty} x(t)\, dt = X(j(0))$$

$$= \lim_{\omega \to 0} \frac{\sin(2\omega)}{\omega}$$

$$= 2$$

2. $$\int_{-\infty}^{\infty} x(t)\, e^{j\frac{\pi}{2}t}\, dt = X\left(j\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{\cos(-2\pi)\ \cancelto{0}{\sin(-\pi)}}{-\pi/2}$$

$$= 0$$

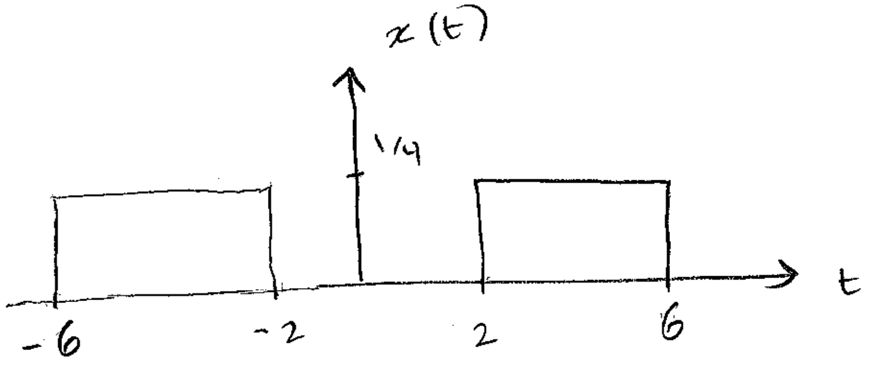
3. $$\frac{\sin(2\omega)}{\omega} \xrightarrow{IFT} \begin{cases} 1/2, & |t| < 2 \\ 0, & |t| > 2 \end{cases} =$$

$$\cos(4\omega) = \frac{e^{4j\omega}}{2} + \frac{e^{-4j\omega}}{2} \xrightarrow{IFT} \frac{1}{2}\delta(t+4) + \frac{1}{2}\delta(t-4)$$

$$\therefore\ x(t) = \begin{cases} 1/4, & |t+4| < 2 \\ 0, & |t+4| > 2 \end{cases} + \begin{cases} 1/4, & |t-4| < 2 \\ 0, & |t-4| > 2 \end{cases}$$

(By convolution property of CT FT)

Plot $\rightarrow$


x(t)
1/4
t
-6
-2
2
6

Question 3: [23%, Work-out question, Learning Objectives 2, 4, and 5] Consider an LTI system for which the input/output relationship is governed by the following differential equation.

$$y(t) + \frac{d}{dt}y(t) = x(t-3) - \frac{d}{dt}x(t-3)$$

We also assume that the LTI system is *initially rest.* That is, if the input is $x(t) = 0$, then the output is $y(t) = 0$.

1. [8%] Find out the frequency response $H(j\omega)$ of this system.
2. [15%] Find out the output $y(t)$ when the input is $x(t) = e^{-2t}\mathcal{U}(t)$.

Hint: If you do not know the $H(j\omega)$ the answer to the first sub-question, you can assume $H(j\omega) = \frac{1}{(1+j\omega)^2}$. You will get full credit for the second ~~and the third sub-questions~~.

1. Taking the CT FT of each side of the equation, we get:

$$Y(j\omega) + \underset{\substack{\uparrow \\ \text{differentiation} \\ \text{in time} \\ \text{property}}}{j\omega\, Y(j\omega)} = \underset{\substack{\uparrow \\ \text{Time-shifting} \\ \text{property}}}{e^{-j3\omega} X(j\omega)} - \underset{\substack{\uparrow \\ \text{Time-shifting property first,} \\ \text{then differentiation} \\ \text{in time property.}}}{j\omega\, e^{-j3\omega} X(j\omega)}$$

$$(1+j\omega)\, Y(j\omega) = \left(e^{-j3\omega} - j\omega\, e^{-j3\omega}\right) X(j\omega)$$

$$\therefore \quad \underset{\substack{\uparrow \\ \text{by} \\ \text{convolution} \\ \text{property}}}{H(j\omega) =} \frac{Y(j\omega)}{X(j\omega)} = \frac{e^{-j3\omega} - j\omega\, e^{-j3\omega}}{1+j\omega}$$

$$= e^{-j3\omega}\, \frac{(1-j\omega)}{1+j\omega}$$

2. $Y(j\omega) = H(j\omega)\, X(j\omega)$

$$X(j\omega) = \frac{1}{2+j\omega}$$

$$Y(j\omega) = \frac{e^{-j3\omega}(1-j\omega)}{(1+j\omega)(2+j\omega)}$$

$$\frac{1-j\omega}{(1+j\omega)(2+j\omega)} = \frac{A}{1+j\omega} + \frac{B}{2+j\omega}$$

$$\Leftrightarrow \quad 1-j\omega = A(2+j\omega) + B(1+j\omega)$$

$$1-j\omega = 2A+B + (A+B)j\omega$$

$$\Rightarrow \begin{cases} 2A+B = 1 \\ A+B = -1 \end{cases} \Rightarrow \begin{array}{l} A = 2 \\ B = -3 \end{array}$$

Therefore, we have that

$$Y(j\omega) = \frac{2e^{-j3\omega}}{1+j\omega} - \frac{3e^{-j3\omega}}{2+j\omega}$$

$$\therefore \quad y(t) = 2e^{-(t-3)}u(t-3) - 3e^{-2(t-3)}u(t-3)$$

Question 4: [17%, Work-out question, Learning Objectives 1, 4, and 5] We know that

$$x(t) = \cos(t) \tag{1}$$

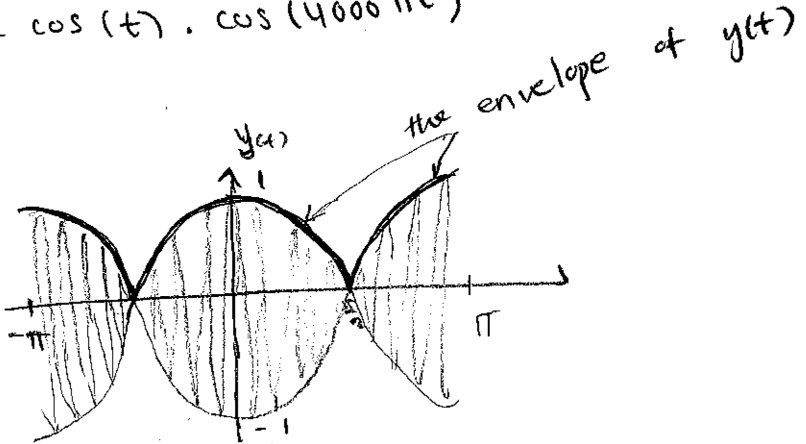
Suppose that we would like to use Amplitude Modulation (AM) to send $x(t)$ with the carrier frequency being 2000Hz. Let $y(t)$ denote the output signal after modulation.

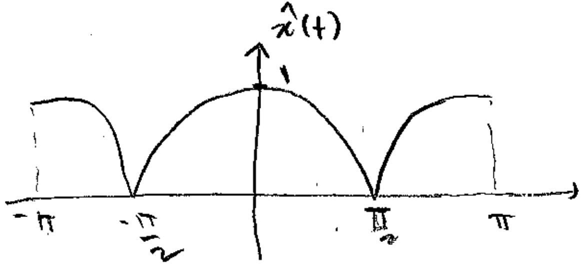
1. [5%] Write down the expression of $y(t)$.
2. [7%] Suppose receiver-1 decides to use *asynchronous demodulation.* Plot the demodulate signal $\hat{x}(t)$ for the range of $-\pi < t < \pi$.
3. [5%] Suppose receiver-2 decides to use *synchronous demodulation.* Let $w(t)$ denote the resulting signal after demodulation. Write down the relationship between $y(t)$ and $w(t)$. Your answer should consist of statements like "multiplying ..." and/or "using a filter ..." Please be specific about the parameters of the filters. If you prefer, you can also use a block diagram (flow chart) to describe your demodulation system instead of using sentences.

1. $y(t) = x(t) \cdot \cos(4000\pi t)$

$= \cos(t) \cdot \cos(4000\pi t)$

2.

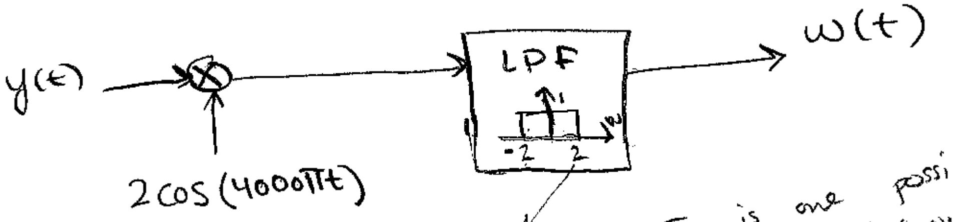




$\hat{x}(t) \approx |x(t)| = |\cos(t)|$

since $4000\pi \gg 1$

3.



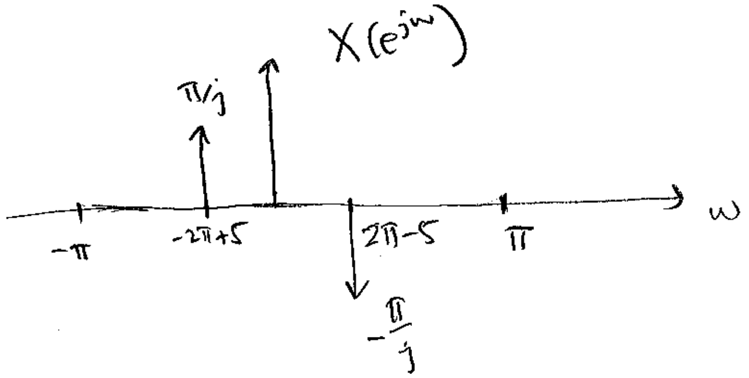
a cut-off freq of 1 rad/sec is also considered correct.

This is one possible synchronous demodulator

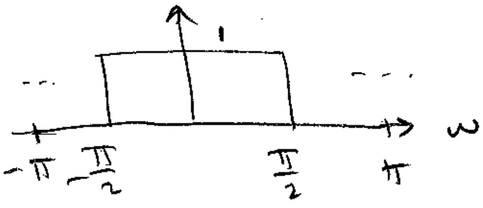
Question 5: [20%, Work-out question, Learning Objectives 4 and 5]

1. [8%] Suppose $x[n] = \sin(5n)$. Plot the discrete-time Fourier transform $X(e^{j\omega})$ for the range of $-\pi < \omega < \pi$.
2. [8%] Suppose $h[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n} e^{-j\pi n}$. Plot the discrete-time Fourier transform $H(e^{j\omega})$ for the range of $-\pi < \omega < \pi$.
3. [4%] Continue from the previous question. Suppose the above $h[n]$ is the impulse response of an LTI system. Is the LTI system a low-pass filter, a band-pass filter, or a high-pass filter? This particular sub-question is a multiple-choice question. No need to justify your answer.

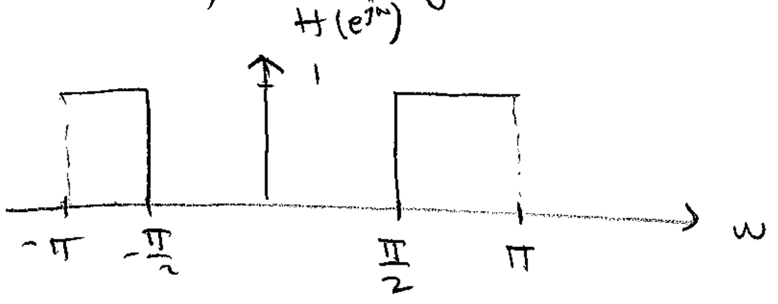
1.



2. $$\frac{\sin(\frac{\pi}{2}n)}{\pi n} \overset{FT}{\longleftrightarrow}$$



By frequency-shifting property



3. A high-pass filter.

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(j\omega)e^{j\omega n}d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi}e^{\sigma t}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{10}$$

Z transform

$$x[n] = r^n\mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{12}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	Ta_kb_k
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [\text{x[n] real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [\text{x[n] real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty}\|x(t)\|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\|X(j\omega)\|^2d\omega$		

ƆRM PAIRS

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TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	**Fourier transform**	**Fourier series coefficients (if periodic)**
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

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TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{Re}\{X(e^{j\omega})\} = \mathcal{Re}\{X(e^{-j\omega})\} \\ \mathcal{Im}\{X(e^{j\omega})\} = -\mathcal{Im}\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{Re}\{X(e^{j\omega})\}$ $j\mathcal{Im}\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational ⇒ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational ⇒ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational ⇒ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—