Final Exam of ECE301, Prof. Wang's section

1–3pm Tuesday, December 11, 2012, Lily 1105.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**

2. This is a closed book exam.

3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.

4. Use the back of each page for rough work.

5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

E-mail:

Signature:

Question 1: [15%, Work-out question]

 1. [1%] What does the acronym "AM-SSB" stands for?

  Prof. Wang wanted to transmit an AM-SSB lower-side-band signal. To that end, he wrote the following MATLAB code.

```
% Initialialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read two different .wav files
[x1, f_sample, N]=wavread('x1');
x1=x1';
[x2, f_sample, N]=wavread('x2');
x2=x2';

% Step 0: Initialize several parameters
W_1=????;
W_2=????;
W_3=????;
W_4=????;
W_5=????;


% Step 1: Make the signals band-limited.
h=1/(pi*t).*(sin(W_1*t));
x1_new=ece301conv(x1, h);
x2_new=ece301conv(x2, h);

% Step 2: Multiply x1_new and x2_new with a sine wave.
x1_h=x1_new.*cos(W_2*t);
x2_h=x2_new.*cos(W_3*t);

% Step 3: Keep the lower side bands
h1=1/(pi*t).*(sin(W_4*t));
h2=1/(pi*t).*(sin(W_5*t));
x1_sb=ece301conv(x1_h, h1);
x2_sb=ece301conv(x2_h, h2);

% Step 4: Create the transmitted signal
y=x1_sb+x2_sb;
wavwrite(y', f_sample, N, 'y.wav');
```

2. [7.5%] Suppose we also know that Prof. Wang intended to use frequency bands 2K–3K Hz and 6K–7K Hz for transmitting x1 and x2, respectively. What should the values of W_1 to W_5 be in the MATLAB code?

Knowing that Prof. Wang used the above code to generate the "y.wav" file, a student tried to demodulate the output waveform "y.wav" by the following code.

```
% Initialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read the .wav files
[y, f_sample, N]=wavread('y');
y=y';

% Initialize several parameters
W_6=????;
W_7=????;
W_8=????;

% Create the low-pass filter.
h_M=1/(pi*t).*(sin(W_6*t));

% Create two band-pass filters.
hBPF_1=1/(pi*t).*(sin(2*pi*3000*t))=1/(pi*t).*(sin(2*pi*2000*t));
hBPF_2=1/(pi*t).*(sin(2*pi*7000*t))=1/(pi*t).*(sin(2*pi*6000*t));


% demodulate signal 1
y1BPF=ece301conv(y,hBPF_1);
y1=4*y1BPF.*sin(W_7*t);
x1_hat=ece301conv(y1,h_M);

sound(x1_hat,f_sample)

% demodulate signal 2
y2BPF=ece301conv(y,hBPF_2);
y2=4*y2BPF.*sin(W_8*t);
x2_hat=ece301conv(y2,h_M);

sound(x2_hat,f_sample)
```

3. [4.5%] Continue from the previous question. What should the values of W_6 to W_8 in the MATLAB code?

Hint: If you do not know the answers to Q1.2 and Q1.3, please simply draw the AMSSB modulation and demodulation diagrams and mark carefully all the parameter values. You will receive 9 points for Q1.2 and Q1.3.

4. [2%] However, even with the correct values of W_6 to W_8, there is still some problem with the above MATLAB code. Please answer (1) What would the student hear using this MATLAB code? (2) How to change the code so that the student can demodulate the signals correct?

Question 2: [16.5%, Work-out question] Consider a continuous-time signal $x(t) = \cos(6\pi t)$, and we sample it with the sampling frequency 2 Hz.

1. [1%] What is the sampling period? (Make sure you write down the correct unit.)

2. [2.5%] Sampling converts the continuous time signal $x(t)$ to a discrete-time array $x[n]$. Plot $x[n]$ for the range of $-2 \le n \le 2$.

3. [4%] We use $x_{\text{lin}}(t)$ to denote the reconstructed signal based on linear interpolation. Plot $x_{\text{lin}}(t)$ for the range of $-2 \le t \le 2$.
   We use $x_{\text{ZOH}}(t)$ to denote the reconstructed signal based on Zero-Order Hold. Plot $x_{\text{ZOH}}(t)$ for the range of $-2 \le t \le 2$.
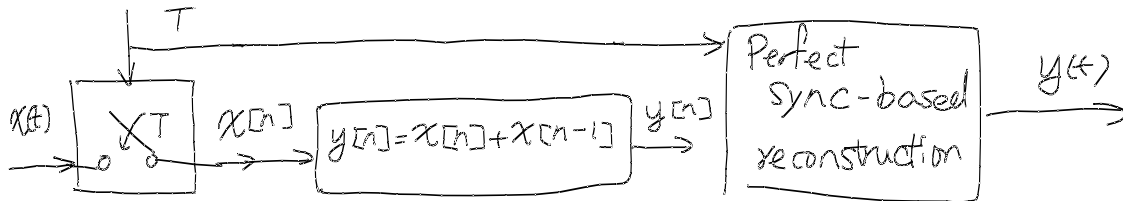
   Hint: If you do not know the answer to Q2.2, you can assume that

   $$x[n] = \begin{cases} 1 & \text{if } n = 4k \text{ for some integer } k \\ -1 & \text{if } n = 4k + 2 \text{ for some integer } k \\ 0 & \text{if } n \text{ is odd} \end{cases} \tag{1}$$

4. [5%] Let $x_p(t)$ denote the impulse-train-sampled signal with the sampling period 2 Hz. Plot $X(j\omega)$ for the range of $-8\pi < \omega < 8\pi$.

5. [4%, advanced] We use $x_{\text{sync}}(t)$ to denote the reconstructed signal based on the optimal reconstruction. Write down the expression of $x_{\text{sync}}(t)$. Plot $x_{\text{sync}}(t)$ for the range of $-2 \le t \le 2$.

   Hint 1: If you do not know the answer to Q2.2, you can use the same assumption as specified in Q2.3.

Question 3: [8%, Work-out question] Consider the following discrete-time signal processing system. Namely, the continuous time input $x(t)$ is sampled first and then processed in discrete-time. In the end, we reconstruct the continuous time output $y(t)$ from the processed array $y[n]$.

Suppose we know that the sampling period is 0.1 second. Determine the end-to-end frequency response $H(j\omega)$ of the above system.

Hint: If you do not know the answer to the above question, you can answer the following two sub-questions instead: Q1: when $x(t) = 1$, what is the output $y(t)$? Q2: when $x(t) = \cos(5\pi t)$, what is the output $y(t)$? You will get 3.5 points and 3 points, respectively, if your answers are correct.

Question 4: [10%, Work-out question]

1. [1%] What is the acronym "ROC" stands for (when considering the Z-transform)?

We know that

$$x[n] = \begin{cases} 2^n & \text{if } n \geq 0 \text{ and } n \text{ is even} \\ 0.2^n & \text{if } n \geq 0 \text{ and } n \text{ is odd} \\ 0 & \text{if } n < 0 \end{cases} \tag{2}$$

2. [9%] Find the Z-transform $X(z)$, write down the expression of the corresponding ROC, and plot the ROC.

   Hint 1: You may need to use the formula: $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$ if $|r| < 1$.

   Hint 2: If you do not know how to solve this question, you can assume

$$x[n] = \begin{cases} 2^n & \text{if } n \geq 0 \\ 3^n & \text{if } n < 0 \end{cases}. \tag{3}$$

   You will get 9 points if your answer is correct.

Question 5: [10%, Work-out question] Consider a continuous-time differential equation system:

$$y(t) = \frac{1}{3}y(t-10) - \frac{1}{2}y(t-20) + x(t) \tag{4}$$

and the system is in a initially rest condition. I.e., when $x(t) = 0$, we have $y(t) = 0$.

Question: When the input is $x(t) = \sum_{k=1}^{6} \left(\frac{1}{2}\right)^k \cos\left(\frac{k\pi}{5}t\right)$, find the corresponding output $y(t)$.

Question 6: [12%, Work-out question] Suppose $x(t) = \frac{2\sin(5t)}{t}$ and $h(t) = \frac{\sin(3t)}{3t} + \cos(20t)$.

1. [4%] Suppose $y(t) = x(t) * h(t)$. Plot $Y(j\omega)$ for the range of $-20 < \omega < 20$.

2. [4%] Suppose $z(t) = x(t) \cdot h(t)$. Plot $Z(j\omega)$ for the range of $-20 < \omega < 20$.

3. [4%] Find $\int_{-\infty}^{\infty} x(t)dt$ and $\int_{-\infty}^{\infty} (x(t))^2 dt$.

Question 7: [12%, Work-out question] Consider two discrete-time signals

$$x[n] = \begin{cases} 3 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 2 \\ \text{periodic with period 3} \end{cases} \tag{5}$$

and

$$h[n] = \frac{\sin\left(\frac{\pi \cdot n}{4}\right)\cos\left(\frac{\pi \cdot n}{4}\right)}{\pi \cdot n}. \tag{6}$$

1. [5%] Find the expression of the DTFT $X(e^{j\omega})$.

2. [4%] Plot the DTFT $H(e^{j\omega})$ for the range of $-4\pi \le \omega \le 4\pi$.

3. [3%] Let $y[n] = x[n] * h[n]$. Find the expression of $y[n]$.

Question 8: [15%, Multiple-choice question] Consider two signals $h_1(t) = \max(\sin(t), \sin(\sqrt{3}t))$ and

$$h_2[n] = \begin{cases} 2^{-n} & \text{if } n \text{ is even and } n \geq 0 \\ 2^{-n-3} & \text{if } n \text{ is odd and } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} \tag{7}$$

1. [1.25%] Is $h_1(t)$ periodic?

2. [1.25%] Is $h_2[n]$ periodic?

3. [1.25%] Is $h_1(t)$ even or odd or neither?

4. [1.25%] Is $h_2[n]$ even or odd or neither?

5. [1.25%] Is $h_1(t)$ of finite energy?

6. [1.25%] Is $h_2[n]$ of finite energy?

Suppose the above two signals are also the impulse responses of two LTI systems: System 1 and System 2, respectively.

1. [1.25%] Is System 1 memoryless?

2. [1.25%] Is System 2 memoryless?

3. [1.25%] Is System 1 causal?

4. [1.25%] Is System 2 causal?

5. [1.25%] Is System 1 stable?

6. [1.25%] Is System 2 stable?

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(j\omega) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1   PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t),\ \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\displaystyle\int_T x(\tau)y(t - \tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\displaystyle\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\dfrac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\dfrac{2\pi}{T}a_k$
Integration		$\displaystyle\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\dfrac{1}{jk\omega_0}\right)a_k = \left(\dfrac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t)\ \text{real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t)\ \text{real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2   PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$ } Periodic with period N and $\;\;\;$ $y[n]$ } fundamental frequency $\omega_0 = 2\pi/N$	a_k } Periodic with $\;\;\;$ b_k } period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\dfrac{1}{m} a_k$ $\begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\displaystyle\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\displaystyle\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\displaystyle\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\dfrac{1}{(1 - e^{-jk(2\pi/N)})} \right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [\text{x[n] real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [\text{x[n] real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $\;\;\;$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1   PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\dfrac{1}{\|a\|}X\left(\dfrac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\dfrac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\displaystyle\int_{-\infty}^{t} x(t)dt$	$\dfrac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\dfrac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\displaystyle\int_{-\infty}^{+\infty} \|x(t)\|^2 dt = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty}\|X(j\omega)\|^2 d\omega$		

TABLE 4.2   BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\displaystyle\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$\displaystyle 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0,$ otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0,$ otherwise
$\sin\omega_0 t$	$\dfrac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0,$ otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1,\quad a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\displaystyle\sum_{k=-\infty}^{+\infty} \dfrac{2\sin k\omega_0 T_1}{k}\,\delta(\omega - k\omega_0)$	$\dfrac{\omega_0 T_1}{\pi}\,\text{sinc}\left(\dfrac{k\omega_0 T_1}{\pi}\right) = \dfrac{\sin k\omega_0 T_1}{k\pi}$
$\displaystyle\sum_{n=-\infty}^{+\infty}\delta(t - nT)$	$\dfrac{2\pi}{T}\displaystyle\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \dfrac{2\pi k}{T}\right)$	$a_k = \dfrac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\dfrac{2\sin\omega T_1}{\omega}$	—
$\dfrac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\dfrac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t),\ \mathcal{R}e\{a\} > 0$	$\dfrac{1}{a + j\omega}$	—
$te^{-at}u(t),\ \mathcal{R}e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^2}$	—
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t),$ $\mathcal{R}e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$	—

TABLE 5.1   PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})$ } periodic with
		$y[n]$	$Y(e^{j\omega})$ } period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n - 1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\displaystyle\sum_{k=-\infty}^{n} x[k]$	$\dfrac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+ \pi X(e^{j0})\displaystyle\sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ |X(e^{j\omega})| = |X(e^{-j\omega})| \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real] $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real]	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\displaystyle\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \dfrac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$		

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

TABLE 5.2   BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\displaystyle\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$\displaystyle 2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$\displaystyle 2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi l)$	(a) $\omega_0 = \dfrac{2\pi m}{N}$ $\quad a_k = \begin{cases} 1, & k=m, m\pm N, m\pm 2N,\ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\dfrac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos\omega_0 n$	$\displaystyle \pi\sum_{l=-\infty}^{+\infty}\{\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)\}$	(a) $\omega_0 = \dfrac{2\pi m}{N}$ $\quad a_k = \begin{cases} \frac{1}{2}, & k=\pm m, \pm m\pm N, \pm m\pm 2N,\ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\dfrac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin\omega_0 n$	$\displaystyle \frac{\pi}{j}\sum_{l=-\infty}^{+\infty}\{\delta(\omega-\omega_0-2\pi l)-\delta(\omega+\omega_0-2\pi l)\}$	(a) $\omega_0 = \dfrac{2\pi r}{N}$ $\quad a_k = \begin{cases} \frac{1}{2j}, & k=r, r\pm N, r\pm 2N,\ldots \\ -\frac{1}{2j}, & k=-r, -r\pm N, -r\pm 2N,\ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\dfrac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n]=1$	$\displaystyle 2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-2\pi l)$	$a_k = \begin{cases} 1, & k=0, \pm N, \pm 2N,\ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n]=\begin{cases} 1, & \lvert n\rvert\le N_1 \\ 0, & N_1<\lvert n\rvert\le N/2 \end{cases}$ and $x[n+N]=x[n]$	$\displaystyle 2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k = \dfrac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N\sin[2\pi k/2N]},\ k\neq 0, \pm N, \pm 2N,\ldots$ $a_k = \dfrac{2N_1+1}{N},\ k=0, \pm N, \pm 2N,\ldots$
$\displaystyle\sum_{k=-\infty}^{+\infty}\delta[n-kN]$	$\displaystyle \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k=\dfrac{1}{N}$ for all k
$a^n u[n],\quad \lvert a\rvert<1$	$\dfrac{1}{1-ae^{-j\omega}}$	—
$x[n]=\begin{cases} 1, & \lvert n\rvert\le N_1 \\ 0, & \lvert n\rvert>N_1 \end{cases}$	$\dfrac{\sin[\omega(N_1+\frac{1}{2})]}{\sin(\omega/2)}$	—
$\dfrac{\sin Wn}{\pi n}=\dfrac{W}{\pi}\,\text{sinc}\left(\dfrac{Wn}{\pi}\right)$ $0<W<\pi$	$X(\omega)=\begin{cases} 1, & 0\le\lvert\omega\rvert\le W \\ 0, & W<\lvert\omega\rvert\le\pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\dfrac{1}{1-e^{-j\omega}}+\displaystyle\sum_{k=-\infty}^{+\infty}\pi\delta(\omega-2\pi k)$	—
$\delta[n-n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n],\quad \lvert a\rvert<1$	$\dfrac{1}{(1-ae^{-j\omega})^2}$	—
$\dfrac{(n+r-1)!}{n!(r-1)!}a^n u[n],\quad \lvert a\rvert<1$	$\dfrac{1}{(1-ae^{-j\omega})^r}$	—

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\dfrac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $|a|R =$ the set of points $\{|a|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., $R^{-1} =$ the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n - 1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $|z| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\dfrac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $|z| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R
10.5.9			Initial Value Theorem	

If $x[n] = 0$ for $n < 0$, then
$$x[0] = \lim_{z \to \infty} X(z)$$

TABLE 10.2   SOME COMMON *z*-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1 - z^{-1}}$	$|z| > 1$
3. $-u[-n - 1]$	$\dfrac{1}{1 - z^{-1}}$	$|z| < 1$
4. $\delta[n - m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1 - \alpha z^{-1}}$	$|z| > |\alpha|$
6. $-\alpha^n u[-n - 1]$	$\dfrac{1}{1 - \alpha z^{-1}}$	$|z| < |\alpha|$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$|z| > |\alpha|$
8. $-n\alpha^n u[-n - 1]$	$\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$|z| < |\alpha|$
9. $[\cos \omega_0 n]u[n]$	$\dfrac{1 - [\cos \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$|z| > 1$
10. $[\sin \omega_0 n]u[n]$	$\dfrac{[\sin \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$|z| > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\dfrac{1 - [r\cos \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$|z| > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\dfrac{[r\sin \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$|z| > r$

10.7.1 Causality

A causal LTI system has an impulse response $h[n]$ that is zero for $n < 0$, and therefore is right-sided. From Property 4 in Section 10.2 we then know that the ROC of $H(z)$ is the exterior of a circle in the z-plane. For some systems, e.g., if $h[n] = \delta[n]$, so that $H(z) = 1$, the ROC can extend all the way in to and possibly include the origin. Also, in general, for a right-sided impulse response, the ROC may or may not include infinity. For example, if $h[n] = \delta[n + 1]$, then $H(z) = z$, which has a pole at infinity. However, as we saw in Property 8 in Section 10.2, for a causal system the power series

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$$

does not include any positive powers of z. Consequently, the ROC includes infinity. Summarizing, we have the follow principle:

A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.