Midterm #2 of ECE301, Prof. Wang's section
8–9pm Thursday, October 13, 2011, EE 170,

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 12 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

E-mail:

Signature:

Question 1: [27%, Work-out question, Outcome 3]

1. [2%] What is an "impulse response"?
2. [10%] Consider a system with the input/output relationship being

$$y(t) = \left(\int_{-\infty}^{t} x(s)e^{-2t+2s}ds\right) + \left(\int_{t}^{\infty} x(s)e^{2t-2s}ds\right). \quad (1)$$

   Find the impulse response $h(t)$ of this system.

3. [15%] Let $z(t) = e^t\mathcal{U}(-t)$. Find the convolution integral $w(t) = z(t) * h(t)$.

   If you do not know the answer $h(t)$ of the previous question, you can assume that $h(t) = \mathcal{U}(t) - \mathcal{U}(-t)$ and solve $w(t)$ accordingly. You will receive 13 points if your answer is correct.

1. Impulse response is the output of the system when the input is an impulse.

2. Let $x(t) = \delta(t)$ then $y(t) = h(t)$

$$h(t) = \int_{-\infty}^{t} \delta(s)\, e^{-2t+2s} ds + \int_{t}^{\infty} \delta(s)\, e^{2t-2s} ds$$

$$= \begin{cases} e^{-2t} & t>0 \\ e^{2t} & t<0 \\ 1 & t=0 \end{cases} = e^{-2|t|}$$

3. $w(t) = z(t) * h(t)$

$$= \int_{\tau=-\infty}^{0} e^{\tau} e^{-2|t-\tau|} d\tau$$

$$= \int_{\tau=-\infty}^{0} e^{\tau} e^{-2(t-\tau)} d\tau \quad \text{if } t>0$$

$$= e^{-2t} \frac{e^{+3\tau}}{3}\Big|_{-\infty}^{0} = e^{-2t}\left(\frac{1}{3} - 0\right) = \frac{e^{-2t}}{3}$$

If $t<0$, $w(t) = \int_{\tau=-\infty}^{t} e^{\tau} e^{-2(t-\tau)} d\tau + \int_{\tau=t}^{0} e^{\tau} e^{-2(\tau-t)} d\tau$

$$= e^{-2t}\int_{-\infty}^{t} e^{3\tau} d\tau + e^{+2t}\int_{\tau=t}^{0} e^{-\tau} d\tau$$

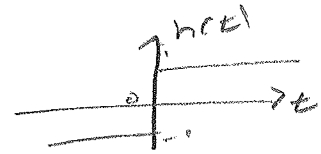
$$= e^{-2t}\left(\frac{1}{3}\right)\left(e^{3t}\right) + \frac{e^{2t}}{-1}\left(1 - e^{-t}\right)$$

$$= \frac{e^t}{3} + e^{2t}(1 - e^{-t})(-1) = \frac{e^t}{3} + e^t - e^{2t}$$

$$= \frac{4}{3} e^t - e^{2t}$$

$$w(t) = \begin{cases} \frac{4}{3} e^t - e^{2t} & \text{if } t < 0 \\ \frac{1}{3} e^{-2t} & \text{if } t > 0 \end{cases}$$

$$h(t) = u(t) - u(-t)$$



$$z(t) = e^t u(-t)$$

$$w(t) = \int_{-\infty}^{0} e^{\tau} \left(u(t-\tau) - u(-(t-\tau)) \right) d\tau$$

If $t > 0$,

$$= \int_{-\infty}^{0} e^{\tau} d\tau = 1$$

If $t \le 0$, $w(t) = \int_{-\infty}^{t} e^{\tau} d\tau + \int_{t}^{0} (-1) e^{\tau} d\tau$

$$= e^t - (1 - e^t) = \underline{\underline{2e^t - 1}}$$

$$w(t) = \begin{cases} 1 & t > 0 \\ 2e^t - 1 & t \le 0 \end{cases}$$

Question 2: [14%, Work-out question, Outcome 2]

1. [2%] What is the acronym "LTI"?

2. [12%] Consider a discrete-time LTI system with its impulse response $h(t)$ being

$$h(t) = \begin{cases} e^{-\sqrt{3}t} & \text{if } 1 \leq t \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

When the input of the system is $x(t) = \cos(\sqrt{3}t) + \sin(t)$, find out the output $y(t)$.

LTI → Linear, Time-Invariant.

$$H(\omega) = \int_{1}^{\infty} e^{-\sqrt{3}t} e^{-j\omega t}\, dt = \int_{+1}^{\infty} e^{-(\sqrt{3}+j\omega)t}\, dt$$

$$= \frac{1}{-(\sqrt{3}+j\omega)} \left(0 - e^{-(\sqrt{3}+j\omega)}\right) = \frac{e^{-(\sqrt{3}+j\omega)}}{\sqrt{3}+j\omega}$$

$$y(t) = \frac{e^{-(\sqrt{3}+j\sqrt{3})}}{\sqrt{3}+j\sqrt{3}} \frac{e^{j\sqrt{3}t}}{2} + \frac{e^{-(\sqrt{3}-j\sqrt{3})}}{\sqrt{3}-j\sqrt{3}} \frac{e^{-j\sqrt{3}t}}{2}$$

(1) (1)

$$+ \frac{e^{-(\sqrt{3}+j)}}{\sqrt{3}+j} \frac{e^{jt}}{2j} - \frac{e^{-(\sqrt{3}-j)}}{\sqrt{3}-j} \frac{e^{-jt}}{2j}$$

(1) (1)

$$= \frac{e^{-\sqrt{3}}}{2\sqrt{6}} e^{-j\pi/4} e^{-j\sqrt{3}} e^{j\sqrt{3}t} + \frac{e^{-\sqrt{3}}}{2\sqrt{6}} e^{j\sqrt{3}} e^{+j\pi/4} e^{-j\sqrt{3}t}$$

$$+ \frac{e^{-\sqrt{3}}}{4j} e^{-j} e^{-j\pi/6} e^{jt} - \frac{e^{-\sqrt{3}}}{4j} e^{+j} e^{+j\pi/6} e^{-jt}$$

$$= \frac{e^{-\sqrt{3}}}{\sqrt{6}} \cos(\sqrt{3}t - \pi/4 - \sqrt{3}) + \frac{e^{-\sqrt{3}}}{2} \sin(t - 1 - \pi/6)$$

Question 3: [25%, Work-out question, Outcome 4]

1. [10%] $y[n] = \cos\left(\frac{6\pi n}{7}\right)$. Find the Fourier series of $y[n]$.
2. [5%] $x(t)$ is periodic with period $T = 4$ and

$$x(t) = \begin{cases} \cos\left(\frac{\pi t}{2}\right) & \text{if } -1 \le t \le 1 \\ 0 & \text{if } 1 \le t < 3 \end{cases}. \qquad (3)$$

   Plot $x(t)$ for the range of $t = -6$ to 6

3. [10%] Compute the Fourier series of $x(t)$.

   Hint: Consider another signal: $z(t)$ is periodic with period $T = 4$ and

$$z(t) = \begin{cases} 1 & \text{if } -1 \le t \le 1 \\ 0 & \text{if } 1 \le t < 3 \end{cases}. \qquad (4)$$

   We know that the Fourier series coefficients of $z(t)$ are

$$b_k = \begin{cases} \frac{1}{2} & \text{if } k = 0 \\ \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} & \text{if } k \neq 0 \end{cases}. \qquad (5)$$

   You may want to use this fact when computing the Fourier series of $x(t)$.

1. $y[n] = \cos\left(\frac{6\pi n}{7}\right) = \frac{e^{j(6\pi n/7)} + e^{-j(6\pi n/7)}}{2}$

$N = 7 \longrightarrow$ period

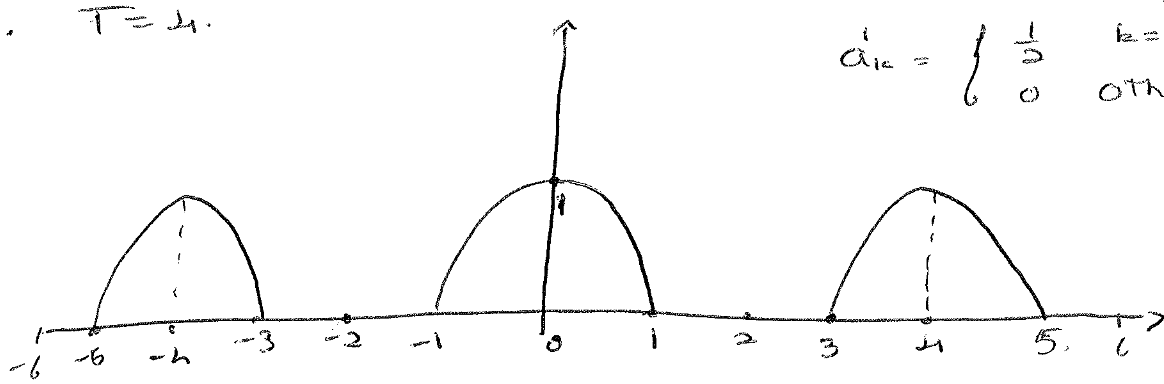
$y[n] = \frac{e^{j\left(\frac{2\pi}{7}\right)(3)}}{2} + \frac{e^{j\left(\frac{2\pi}{7}\right)(-3)}}{2}$

$\Rightarrow a_k = \begin{cases} \frac{1}{2} & k = 3, -3 \\ 0 & \text{otherwise.} \end{cases}$

$\cos\left(\frac{\pi t}{2}\right) = \frac{e^{j\frac{2\pi}{4}(1)t} + e^{-j\frac{2\pi}{4}t}}{2}$

$a_k = \begin{cases} \frac{1}{2} & k = 1, -1 \\ 0 & \text{otherwise.} \end{cases}$

2. $T = 4$.



CURVE - 4, VALUES - 1

$\Rightarrow a_k = a_k' \cdot b_k$

$$a_k = \frac{1}{4}\left[\int_{-1}^{1} \cos\left(\frac{\pi t}{2}\right) e^{-j\frac{2\pi}{4}kt}\right]$$

$$= \frac{1}{4}\int_{-1}^{1} \frac{e^{j\frac{\pi t}{2}} + e^{-j\frac{\pi t}{2}}}{2}\, e^{-j\frac{\pi}{2}kt}\, dt$$

$$= \frac{1}{8}\left[\int_{-1}^{1} e^{j\frac{\pi}{2}t(1-k)}\, dt + \int_{-1}^{1} e^{-j\frac{\pi}{2}t(1+k)}\, dt\right]$$

$$= \frac{1}{8}\left[\frac{1}{j\frac{\pi}{2}(1-k)}\left(2j\sin\left(\frac{\pi}{2}(1-k)\right)\right) + \frac{1}{-j\frac{\pi}{2}(1+k)}\left(-2j\sin\left(\frac{\pi}{2}(1+k)\right)\right)\right]$$

$$= \frac{1}{2\pi(1-k)}\sin\left(\frac{\pi(1-k)}{2}\right) + \frac{1}{2\pi(1+k)}\sin\left(\frac{\pi}{2}(1+k)\right)$$

AND

$$= \frac{1}{4} + \frac{1}{4}\sin(\pi) = \frac{1}{4} \qquad k = 1$$

$$k = -1$$

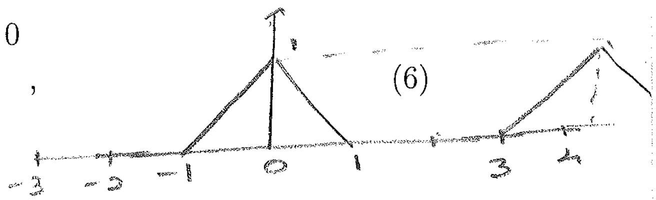
AND $= \frac{\sin(\pi)}{\pi(4)} + \frac{1}{4} = 1/4$

Using properties.

$$a_k = \frac{b_{k-1} + b_{k+1}}{2} = \begin{cases} \frac{1}{4}, & k = 1, -1 \\ \frac{1}{2}\frac{\sin((k-1)\pi/2)}{(k-1)\pi} + \frac{1}{2}\frac{\sin((k+1)\pi/2)}{(k+1)\pi} \end{cases}$$

Question 4: [14%, Work-out question, Outcome 4] We know that if $x(t)$ is periodic with period $T = 4$ and

$$x(t) = \begin{cases} t+1 & \text{if } -1 \le t < 0 \\ 1-t & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < 3 \end{cases}, \tag{6}$$

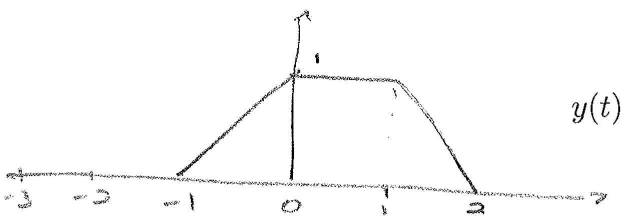


then the corresponding FS coefficients are

$$a_k = \begin{cases} \frac{1}{4} & \text{if } k = 0 \\ \frac{2-2\cos\left(\frac{k\pi}{2}\right)}{k^2\pi^2} & \text{if } k \neq 0 \end{cases}. \tag{7}$$

Answer the following questions:

1. [7%] Compute the value of $\sum_{k=-\infty}^{\infty} a_k^2$.
2. [7%] Consider a different signal $y(t)$, which is periodic with period $T = 4$ and



$$y(t) = \begin{cases} t+1 & \text{if } -1 \le t < 0 \\ 1 & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < 2 \\ 0 & \text{if } 2 \le t < 3 \end{cases}. \tag{8}$$

Find the Fourier series of $y(t)$.

$T = 4.$ $\quad \sum_{k=-\infty}^{\infty} a_k^2 = \frac{1}{4}\left[\int_{-1}^{0} (t+1)^2 dt + \int_{0}^{1} (1-t)^2 dt + \int_{1}^{3} 0\, dt\right]$

$= \frac{1}{4}\left[\frac{(t+1)^3}{3}\Big|_{-1}^{0} + \frac{(1-t)^3}{-3}\Big|_{0}^{1} + 0\right]$

$= \frac{1}{4}\left[\frac{1}{3}(1-0) - (0 - \frac{1}{3})\right] = \frac{1}{4}(2/3)$

$= 1/6$

2. $T = 4.$ $\quad a_k = \frac{1}{4}\left[\int_{-1}^{0} (t+1) e^{-j\frac{\pi}{2}kt} dt + \int_{0}^{1} e^{-j\pi/2 kt} dt + \int_{1}^{2} (2-t) e^{-j\pi/2 kt} dt\right]$

$= \frac{1}{4}\Big[$

$$= \frac{1}{4}\left[\frac{(t+1)e^{-j\frac{\pi}{2}kt}}{-j\frac{\pi}{2}k}\Bigg|_{-1}^{0} - \int_{-1}^{0} \frac{e^{-j\frac{\pi}{2}kt}}{-j\frac{\pi}{2}k}dt \right.$$

$$+ \frac{1}{-j\frac{\pi}{2}k}\left(e^{-j\frac{\pi}{2}k}-1\right) + \frac{(2-t)e^{-j\frac{\pi}{2}kt}}{-j\frac{\pi}{2}k}\Bigg|_{1}^{2}$$

$$\left. + \int_{1}^{2} \frac{e^{-j\frac{\pi}{2}kt}}{-j\frac{\pi}{2}k}dt \right]$$

$$= \frac{1}{4}\left[\frac{-2}{j\pi k}(1-0) + \frac{4}{\pi^2k^2}\left(1-e^{j\frac{\pi}{2}k}\right) \right.$$

$$+ \frac{-2}{j\pi k}\left(e^{-j\frac{\pi}{2}k}-1\right) + \left(0-e^{-j\frac{\pi}{2}k}\right)\frac{-2}{j\pi k}$$

$$\left. + \frac{-4}{\pi^2k^2}\left(e^{-j\pi k}-e^{-j\frac{\pi}{2}k}\right) \right]$$

$$= \frac{1}{\pi^2k^2}\left(1-(-1)^k-2j\sin\left(\frac{\pi}{2}k\right)\right)$$

$$a_0 = \frac{1}{4}\left[\int_{-1}^{0}(t+1)dt + \int_{0}^{1}1\,dt + \int_{1}^{2}(2-t)dt\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}(1) + 1 + \frac{-1}{2}(0-1)\right) = \frac{1}{2}$$

OR

$$b_k = a_k\left(1+e^{-jk\frac{2\pi}{4}(1)}\right) = \begin{cases} \frac{1}{2} & \text{if } k=0 \\ \frac{2-2\cos\frac{k\pi}{2}}{(k\pi)^2}\left(1+(-j)^k\right) & \text{if } k\neq 0 \end{cases}$$

because $y(t) = x(t) + x(t-1)$

Question 5: [20%, Multiple Choices] The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \begin{cases} \int_{-\infty}^{t} x_1(2s)ds & \text{if } 0 \le t \\ 0 & \text{if } t < 0 \end{cases}. \tag{9}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = \min(0, x_2[n+2]). \tag{10}$$

Answer the following questions

1. [4%, Outcome 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Outcome 1] Is System 1 causal? Is System 2 causal?
3. [4%, Outcome 1] Is System 1 stable? Is System 2 stable?
4. [4%, Outcome 1] Is System 1 linear? Is System 2 linear?
5. [4%, Outcome 1] Is System 1 time-invariant? Is System 2 time-invariant?

1.	NO /	NO	WM	WM
2.	NO /	NO	NC	NC
3.	NO /	YES	NS	S
4.	YES /	NO	L	NL
5.	NO /	YES	TV	TI

$$\int_{-\infty}^{t} x_1(2s - t_0)\, ds$$

$$2z = 2s - t_0$$

$$\int_{-\infty}^{(2t-t_0)/2} x_1(2z)\, dz$$