Question 1: [20%, Work-out question, Outcome 3] $x(t)$ is $e^{(1+j)t}$ if $-2 \le t \le 1$, and $x(t) = 0$ otherwise. Compute the expression of

$$y(t) = \int_{-\infty}^{\infty} x(s)x(t-s)ds.$$

$$x(t) = \begin{cases} e^{(1+j)t} & -2 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t-s) = \begin{cases} e^{(1+j)(t-s)} & -2 \le t-s \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{(1+j)(t-s)} & -2-t \le -s \le 1-t \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e^{(1+j)(t-s)} & 2+t \ge s \ge t-1 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \int_{-2}^{1} e^{(1+j)s} x(t-s)\, ds$$

If $t+2 < -2 \Rightarrow t < -4$, $y(t) = 0$.

If $t-1 < -2 \Rightarrow t < -1$, and $2+t < 1$

$$y(t) = \int_{-2}^{t+2} e^{(1+j)s} e^{(1+j)(t-s)} ds$$

$$= \int_{-2}^{t+2} e^{(1+j)t} ds = e^{(1+j)t}(t+4)$$

If $t-1 \ge -2 \Rightarrow t \ge -1$ and $2+t \ge 1$

$$y(t) = \int_{t-1}^{1} e^{(1+j)s} e^{(1+j)(t-s)} ds = e^{(1+j)(t)}(2-t)$$

If $t-1 \ge 1 \Rightarrow t \ge 2$, $y(t) = 0$.

$$y(t) = \begin{cases} 0 & t < -4 \\ e^{(1+j)t}(t+4) & -4 \le t \le -1 \\ e^{(1+j)t}(2-t) & -1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

Question 2: [15%, Work-out question, Outcome 4] $x[n] = 1$ if $10 \leq n \leq 30$, and $x[n] = 0$ otherwise. Compute the expression of

$$g(\omega) = \sum_{n=-\infty}^{\infty} x[2-n]e^{j\omega n}.$$

Hint: The geometric series formula is

$$\sum_{k=1}^{K} a_1 r^{k-1} = \frac{a_1(1-r^K)}{1-r}.$$

$$x[n] = \begin{cases} 1 & 10 \leq n \leq 30 \\ 0 & \text{otherwise} \end{cases} \Rightarrow x[2-n] = \begin{cases} 1 & 10 \leq 2-n \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow x[2-n] = \begin{cases} 1 & -8 \geq n \geq -28 \\ 0 & \text{otherwise.} \end{cases}$$

$$g(\omega) = \sum_{n=-\infty}^{\infty} x[2-n]\, e^{j\omega n}$$

$$= \sum_{n=-28}^{-8} e^{j\omega n} = \sum_{k=1}^{21} e^{j\omega(k-29)}$$

$$(\because k = n+29) \qquad = e^{-j\omega(29)} \sum_{k=1}^{21} e^{j\omega k} \cdot \frac{e^{-j\omega}}{e^{-j\omega}}$$

$$\frac{e^{-8j\omega} - e^{-29j\omega}}{1-e^{-j\omega}}$$

$$= e^{-j\omega 28} \sum_{k=1}^{21} e^{j\omega(k-1)}$$

$$= e^{-28j\omega} \frac{(1-e^{j\omega(21)})}{1-e^{j\omega}}$$

$$\frac{e^{-28j\omega} - e^{-7j\omega}}{1-e^{j\omega}}$$

$$= \frac{e^{-28j\omega}(1-e^{21j\omega})}{1-e^{j\omega}}$$

Question 3: [10%, Work-out question, Outcome 4] Compute the following partial fractions:

$$\frac{1}{-2+3z+5z^2} = \frac{a}{b+c\cdot z} + \frac{d}{e+f\cdot z}.$$

Namely, find the real-valued coefficients a, b, c, d, e, and f.

Compute the following partial fractions:

$$\frac{1}{-2+3z+5z^2} \times \frac{1-z}{1+z} = \frac{a}{b+c\cdot z} + \frac{d}{e+f\cdot z} + \frac{g}{(b+c\cdot z)^2}.$$

Namely, find the real-valued coefficients a, b, c, d, e, f, and g.

$$5z^2+3z-2 = 5z^2+5z-2z-2$$
$$= 5z(z+1) - 2(z+1) = (5z-2)(z+1)$$

$$\frac{1}{-2+3z+5z^2} = \frac{a}{5z-2} + \frac{d}{z+1}$$

$\Rightarrow 1 = a(z+1) + d(5z-2)$ — ①

If $z=-1$, $1 = d(-7) \Rightarrow d = -1/7$

If $z = 2/5$, $1 = a(7/5) + d(0) \Rightarrow a = 5/7$

$$\Rightarrow \frac{1}{-2+3z+5z^2} = \frac{5}{7(5z-2)} + \frac{-1}{7(z+1)}$$

$\Rightarrow a = 5$, $b = -14$, $c = 35$, $d = -1$, $e = 7$, $f = 7$

$a = 5/7$, $b = -2$, $c = 5$, $d = -1/7$, $e = 1$, $f = 1$.

$$\frac{1-z}{(5z-2)(1+z)^2} = \frac{a}{1+z} + \frac{d}{5z-2} + \frac{g}{(1+z)^2}$$

$\Rightarrow 1-z = a(5z-2)(1+z) + d(1+z)^2 + g(5z-2)$

If $z=-1$, $2 = a(0) + d(0) + g(-7) \Rightarrow g = -2/7$

If $5z=2$, $1-2/5 = a(0) + d(1+2/5)^2 + g(0)$

$\Rightarrow 3/5 = d(7/5)^2 \Rightarrow d = \frac{3\times5}{49} = \frac{15}{49}$

$0 = 5a + d \Rightarrow a = -d/5 = -\frac{15}{5\times49} = -\frac{3}{49}$

$$\frac{1-z}{(5z-2)(1+z)^2} = \frac{-3}{49(1+z)} + \frac{15}{49(5z-2)} + \frac{-2}{7(1+z)^2}$$

$\Rightarrow a = -3/49$, $b=1$, $c=1$, $d = 15/49$, $f=5$, $e=-2$

$g = -2/7$, $b=1$, $c=1$

Question 4: [15%, Work-out question, Outcome 1] We know that $x(t)$ is a continuous-time period signal with period 3. We also know that

$$x(t) = \begin{cases} t+1 & \text{if } -1 \le t < 0 \\ \cos(2t) + j\sin(2t) & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < 2 \end{cases} \tag{1}$$

Find the *overall power* of this signal.

Energy over a period $= \int x^2(t)\,dt$

$$= \int_{-1}^{0} (t+1)^2 dt + \int_{0}^{1} (1)\,dt + \int_{1}^{2} (0)\,dt \qquad \to \cos^2(2t) + \sin^2(2t)$$

$$= \left.\frac{(t+1)^3}{3}\right|_{-1}^{0} + t\Big|_0^1 = \frac{1}{3}(1-0) + 1$$

$$= 1 + 1/3 = 4/3.$$

$\Rightarrow$ Power $= \frac{\text{Energy}}{\text{Period}} = \frac{4/3}{3} = \underline{4/9}$ ✓

Question 5: [20%, Work-out question, Outcome 1] Consider a linear system. We know that for this system if the input $x(t)$ is an even signal, then the output $y(t)$ is always zero, that is, $y(t) = 0$. We also know that if the input $x(t)$ is an odd signal, the output $y(t)$ is always $y(t) = x(t-2)$.

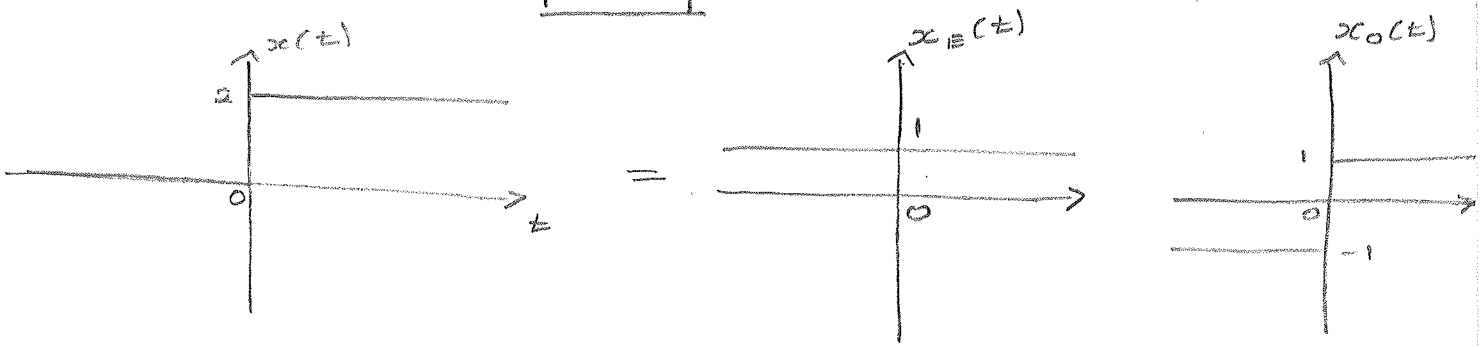
Question: Plot the output $y(t)$ for the range $-4 \leq t \leq 4$ when the input is

$$x(t) = \begin{cases} 2 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}. \tag{2}$$

If you do not know the answer to the above question, you can assume $x(t) = \sin(2t) + j\cos(3t)$ and plot the corresponding $y(t)$. You will get still get 14 points if your answer is correct.

$$x(t) \xrightarrow{\text{Even}} \boxed{H} \longrightarrow y(t) = 0$$

$$x(t) \xrightarrow{\text{odd}} \boxed{H} \longrightarrow y(t) = x(t-2)$$

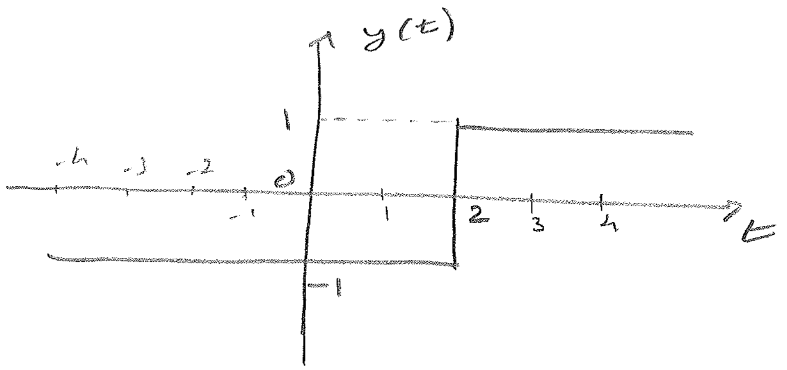


$\Rightarrow x(t) = x_E(t) + x_o(t)$

$\Rightarrow y(t) = y_E(t) + y_o(t)$ (linearity)

$= 0 + y_o(t)$

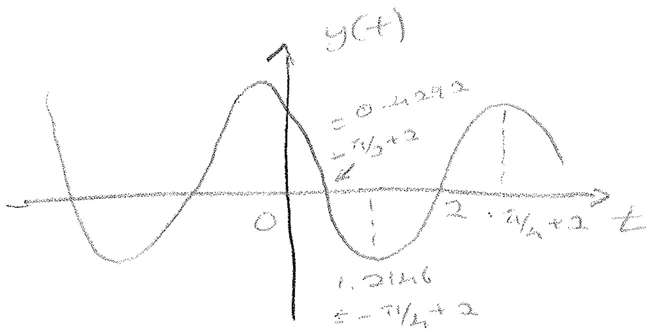
$= x_o(t-2)$



If $x(t) = \sin(2t) + j\cos(3t)$

$y(t) = \sin(2(t-2))$

$= \sin(2t-4)$



Question 6: [20%, Multiple Choices] The following questions are multiple-choice questions and there is no need to justify your answers. Consider two continuous-time signals:

$$x_1(t) = e^{(\cos(t)+j\sin(t))}$$
$$x_2(t) = \frac{\sin(t)}{e^t + e^{-t}}$$

$e^{(\cos(t+T))} + e^{j(\sin(t+T))}$

let $T = 2\pi$

$= e^{\cos t} \; e^{j\sin t}$

and two discrete-time signals:

$$x_3[n] = \cos(\pi n^2) + j\sin(\pi n)$$
$$x_4[n] = e^{(1+j)n}.$$

$\cos(\pi(n^2 + N^2 + 2nN)) + j\sin(\pi(n+N))$

$N = 2$

$= \cos(\pi n^2) + j\sin(\pi n)$

1. [10%, Outcome 1] For $x_1(t)$ to $x_4[n]$, determine whether it is periodic or not, *respectively*. If it is periodic, write down the fundamental period. Please state explicitly which signal is periodic and which is not.

2. [10%, Outcome 1] For $x_1(t)$ to $x_4[n]$, determine whether it is even or odd or neither of them, respectively. Please state explicitly which signal is even, which is odd, and which is neither.

(1) $x_1(t) \rightarrow$ periodic. $T = 2\pi$

$x_2(t) \rightarrow$ non-periodic

$x_3[n] \rightarrow$ periodic, $N = 2$

$x_4[n] \rightarrow$ non-periodic

(2) $x_1(t) \rightarrow$ neither

$x_2(t) \rightarrow$ odd

$x_3[n] \rightarrow$ Even

$x_4[n] \rightarrow$ neither