ECE 301, Midterm #3

6:30-7:30pm Thursday, April 15, 2010, LYNN 1136,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 16 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

E-mail:

Signature:

Question 1: [30%, Work-out question] Consider a continuous time signal $x(t)$:

$$x(t) = \mathcal{U}(t-\pi) - \mathcal{U}(t+\pi). \tag{1}$$

1. [10%, Outcomes 1, 5] Find the expression and plot the FT $X(j\omega)$ of $x(t)$ for the range of $-3 < \omega < 3$. Carefully mark the height of the main lobe and the points when $X(j\omega) = 0$. Hint: Use the table.

2. [10%, Outcomes 3, 4] Define $h(t) = 2\delta(t-\pi) + 3\delta(t+\pi)$. Let $y(t) = x(t) * h(t)$. Plot $y(t)$ for the range of $-3\pi < t < 3\pi$.

3. [10%, Outcome 5] Find the FT $H(j\omega)$ of $h(t)$. Hint: No need to simplify it to sin or cos functions.

1.

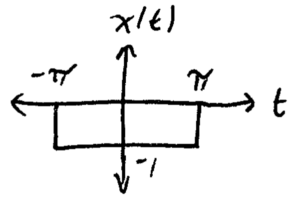


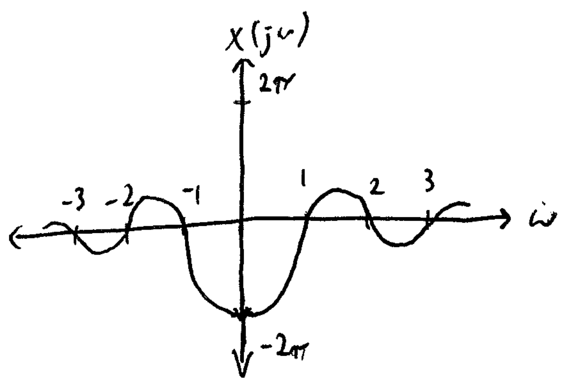
Table : $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \Longleftrightarrow \dfrac{2\sin \omega T_1}{\omega}$

$$X(j\omega) = \frac{-2\sin(\pi\omega)}{\omega}$$

$$X(j0) = \frac{-2\pi\cos(\pi 0)}{1} = -2\pi$$

$$X(j\omega) \Rightarrow \sin(\pi\omega) = 0$$

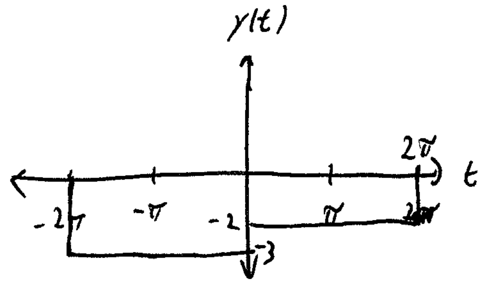
$$\Rightarrow \omega = k \qquad k \in \mathbb{Z}$$



2. $h(t) = 2\delta(t-\pi) + 3\delta(t+\pi)$

$y(t) = x(t) * h(t)$

$= 2x(t-\pi) + 3x(t+\pi)$



3. $H(j\omega) = 2e^{-j\pi\omega} + 3e^{j\pi\omega}$

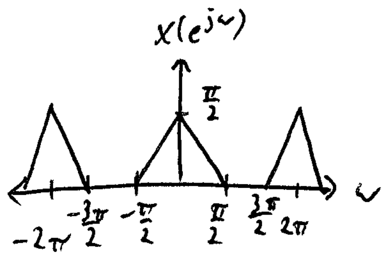
Question 2: [35%, Work-out question, Outcome 4] Consider a discrete time signal $x[n]$ and its Fourier transform $X(e^{j\omega})$. Suppose we know that within the range of $-\pi < \omega < \pi$, $X(e^{j\omega})$ can be described as follows.

$$X(e^{j\omega}) = \begin{cases} \omega + \pi/2 & \text{if } -\frac{\pi}{2} < \omega < 0 \\ -\omega + \pi/2 & \text{if } 0 < \omega < \frac{\pi}{2} \\ 0 & \text{if } -\pi < \omega < -\frac{\pi}{2} \text{ or if } \frac{\pi}{2} < \omega < \pi \end{cases} . \tag{2}$$

The following questions are best done in sequence. However, you can also do them separately if you do not know the answers to some of the questions.

1. [5%, Outcome 5] Plot $X(e^{j\omega})$ for the range $-2\pi < \omega < 2\pi$.
2. [5%, Outcome 5] Define $Y(e^{j\omega}) = \frac{d}{d\omega}X(e^{j\omega})$. Plot $Y(e^{j\omega})$ for the range $-2\pi < \omega < 2\pi$.
3. [10%, Outcomes 4, 5] Find the expression of $y[n]$. Hint 1: First consider a rectangular wave form in frequency. Hint 2: It can then be solved by DTFT properties.
4. [9%], Outcome 4 Find the value of $x[0]$. This problem can still be solved even if you do not know the answer to sub-questions 2 and 3.
5. [6%], Outcome 4 Find the expression of $x[n]$ for $n \neq 0$. This problem can still be solved even if you do not know the answer to sub-questions 2 and 3.

1.



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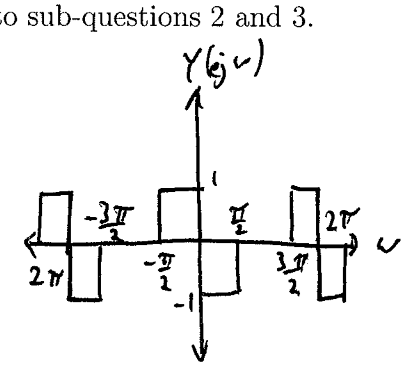

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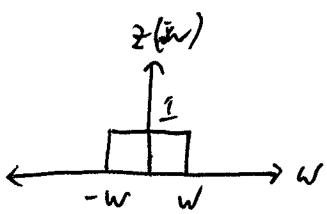


$\Longleftrightarrow \quad \frac{\sin(Wn)}{\pi n} = Z[n] \qquad W = \frac{\pi}{4}$

$Y(e^{j\omega}) = Z(e^{j(\omega+\frac{\pi}{4})}) - Z(e^{j(\omega-\frac{\pi}{4})}) \Leftrightarrow y[n] = e^{-j\frac{\pi}{4}n}Z[n] - e^{+j\frac{\pi}{4}n}Z[n]$

$y[n] = -2j\sin(\frac{\pi}{4}n)\frac{\sin(\frac{\pi}{4}n)}{\pi n} = -j\frac{2\sin^2(\frac{\pi}{4}n)}{\pi n}$

4. $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$

$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,d\omega = \frac{1}{2\pi}\left[\frac{1}{2}\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)\right]$

$= \frac{1}{2\pi}\left[\frac{\pi^2}{4}\right] = \frac{\pi}{8}$

5. $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$

$= \frac{1}{2\pi}\int_{-\pi/2}^{0} (\omega+\frac{\pi}{2})e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}} (-\omega+\frac{\pi}{2})e^{j\omega n}\,d\omega$

$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{2}e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{0} \omega e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}} -\omega e^{j\omega n}\,d\omega$

$u = \omega \qquad dv = e^{j\omega n}\,d\omega$

$du = d\omega \qquad v = \frac{1}{jn}e^{j\omega n}$

$= \frac{1}{2\pi}\left(\frac{\pi}{2}\right)\frac{1}{jn}e^{j\omega n}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{2\pi}\left[\frac{\omega}{jn}e^{j\omega n}\Big|_{-\frac{\pi}{2}}^{0} - \int_{-\frac{\pi}{2}}^{0} \frac{1}{jn}e^{j\omega n}\,d\omega\right]$

$- \frac{1}{2\pi}\left[\frac{\omega}{jn}e^{j\omega n}\Big|_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} \frac{1}{jn}e^{j\omega n}\,d\omega\right]$

$= \frac{1}{4jn}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right) + \frac{1}{2\pi}\frac{1}{jn}\left(+\frac{\pi}{2}e^{-j\frac{\pi}{2}n} - \frac{\pi}{2}e^{j\frac{\pi}{2}n}\right)$

$- \frac{1}{2\pi}\frac{1}{(jn)^2}e^{j\omega n}\Big|_{-\frac{\pi}{2}}^{0} + \frac{1}{2\pi}\frac{1}{(jn)^2}e^{j\omega n}\Big|_{0}^{\frac{\pi}{2}}$

$= \frac{1}{jn}\left(\frac{1}{4}e^{j\frac{\pi}{2}n} - \frac{1}{4}e^{-j\frac{\pi}{2}n} + \frac{1}{4}e^{-j\frac{\pi}{2}n} - \frac{1}{4}e^{j\frac{\pi}{2}n}\right)$

$+ \frac{1}{2\pi n^2}\left[+\left(1 - e^{-j\frac{\pi}{2}n}\right) - \left(e^{j\frac{\pi}{2}n} - 1\right)\right]$

$= \frac{-1}{2\pi n^2}\left[e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n} - 2\right] = \frac{-\cos\left(\frac{\pi}{2}n\right)}{\pi n^2} + \frac{1}{\pi n^2} = \frac{1-\cos\left(\frac{\pi}{2}n\right)}{\pi n^2}$

5. $y[n] = \frac{1}{j}n\,x[n] \Rightarrow x[n] = \frac{j}{n}y[n] = \frac{+2\sin^2\left(\frac{\pi}{4}n\right)}{\pi n^2}$

Note: $\sin^2\left(\frac{\pi}{4}n\right) = \left(\frac{1}{2j}\right)^2\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)^2 = -\frac{1}{4}\left(2\cos\left(\frac{\pi}{2}n\right) - 2\right)$

Question 3: [20%, Work-out question] Consider a discrete time LTI system. We know that when the input is

$$x[n] = \left(\frac{1}{2}\right)^n \mathcal{U}[n],$$

the output is

$$y[n] = x[n] * h[n] = 8\left(\frac{4}{5}\right)^n \mathcal{U}[n] - 5\left(\frac{1}{2}\right)^n \mathcal{U}[n].$$

1. [10%, Outcomes 2, 4, 5] Find out the impulse response $h[n]$ of this system.
2. [10%, Outcomes 2, 4, 5] When a new input $w[n] = \cos(n)$ is used, find the new output $z[n] = w[n] * h[n]$. If you do not know the answer to the previous sub-question, you can assume $h[n] = \mathcal{U}[n+4] - \mathcal{U}[n-5]$. You will still get full credit if your answer is right.

1. $X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$ $\quad Y(e^{j\omega}) = \frac{8}{1-\frac{4}{5}e^{-j\omega}} - \frac{5}{1-\frac{1}{2}e^{-j\omega}}$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \left(\frac{8}{1-\frac{4}{5}e^{-j\omega}} - \frac{5}{1-\frac{1}{2}e^{-j\omega}}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)$$

$$= \frac{8-4e^{-j\omega}}{1-\frac{4}{5}e^{-j\omega}} - 5 = \frac{8-4e^{-j\omega}-5+4e^{-j\omega}}{1-\frac{4}{5}e^{-j\omega}}$$

$$= \frac{3}{1-\frac{4}{5}e^{-j\omega}} \Rightarrow h[n] = 3\left(\frac{4}{5}\right)^n \mathcal{U}[n]$$

2. $w[n] = \frac{1}{2}e^{jn} + \frac{1}{2}e^{-jn}$

$$\Rightarrow z[n] = \frac{1}{2}H(e^{j(1)})e^{jn} + \frac{1}{2}H(e^{j(-1)})e^{-jn}$$

$$= \frac{1}{2}\frac{3}{1-\frac{4}{5}e^{-j}}e^{jn} + \frac{1}{2}\frac{3}{1-\frac{4}{5}e^{j}}e^{-jn}$$

$$= \frac{3}{2}\left[\frac{1-\frac{4}{5}e^{j}}{1+\frac{16}{25}-\frac{8}{5}\cos(1)}e^{jn} + \frac{1-\frac{4}{5}e^{-j}}{1+\frac{16}{25}-\frac{8}{5}\cos(1)}e^{-jn}\right]$$

2. (Alt)

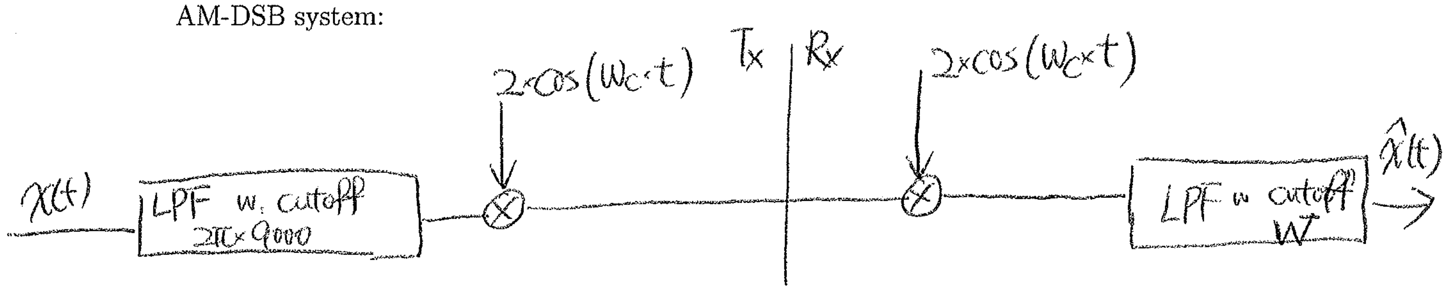
If $h[n] = u[n+4] - u[n-5] = \begin{cases} 1 & |n| \le 4 \\ 0 & \text{ow} \end{cases}$

$$H(e^{j\omega}) = \frac{\sin(\omega(4+\frac{1}{2}))}{\sin(\frac{\omega}{2})} = \frac{\sin(\frac{9}{2}\omega)}{\sin(\frac{\omega}{2})}$$

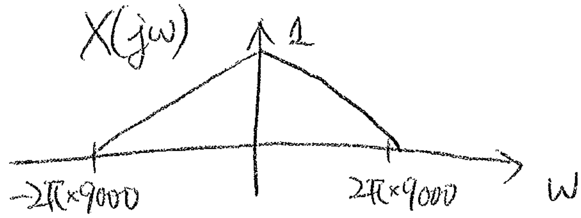
$$z[n] = \frac{1}{2} H(e^{j(1)}) e^{jn} + \frac{1}{2} H(e^{j(-1)}) e^{-jn}$$

$$= \frac{1}{2} \frac{\sin(+\frac{9}{2})}{\sin(\frac{1}{2})} e^{jn} + \frac{1}{2} \frac{\sin(-\frac{9}{2})}{\sin(-\frac{1}{2})} e^{-jn}$$

Question 4: [15%. No need to write down any explanation] Consider the following AM-DSB system:



We know that the input $x(t)$ has bandwidth 9kHz (or equivalently $9k \cdot 2\pi$ radian per second), i.e., $X(j\omega) = 0$ for all $|\omega| > 2\pi \cdot 9000$. If you like, you can also assume that $X(j\omega)$ is described as follows.



Suppose the FCC requires that this radio transmission can only use the frequency band between 2.4MHz to 2.43MHz and must not use any bandwidth outside the given 2.4–2.43MHz range. Answer the following questions.

1. [5%, Outcomes 3, 4, 5] What is the allowable range of the carrier frequency ω_c?
2. [5%, Outcomes 3, 4, 5] Suppose we use ω_c =2.41MHz. What is the allowable range of the cutoff frequency W of the low-pass filter at the receiver, assuming that $x(t)$ is the only signal being transmitted in the air?
3. [5%, Outcomes 3, 4, 5] If we listen to the output signal $\hat{x}(t)$, is the volume of the sound going to be louder, the same, or weaker when compared to listening to the original signal $x(t)$.

1. $x(t)$ is band-limited to $\pm$ 9kHz (0.009 MHz)

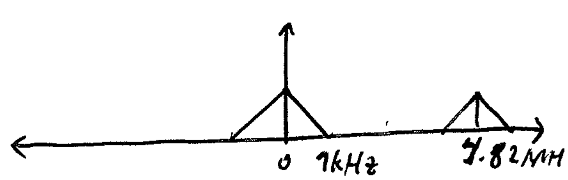
$f_c \geq 2.4$ MHz + 9kHz = 2.409 MHz

$f_c \leq 2.43$ MHz − 9kHz = 2.421 MHz

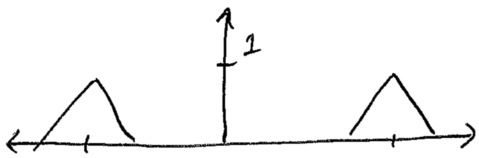
2.409 MHz $\leq \frac{\omega_c}{2\pi} \leq$ 2.421 MHz

2. $W \geq$ 9kHz or it will cut out some of the signal

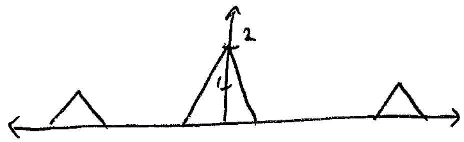
$w \leq 2(2.41$ MHz$)$ − 9kHz = 4.82 − 0.009 MHz = 4.811 MHz



3.

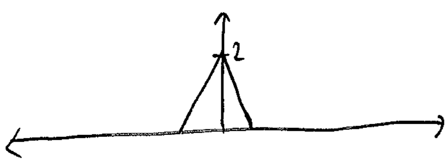


at the transmitter



at the receiver after multiplying by $2\cos(\omega_c t)$

$\tilde{X}(j\omega$



The magnitude of $\tilde{x}(t)$ is twice that of $x(t)$, and it will therefore be louder.