ECE 301, Midterm #2

6:30-7:30pm Thursday, February 25, 2010, LYNN 1136,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 11 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

E-mail:

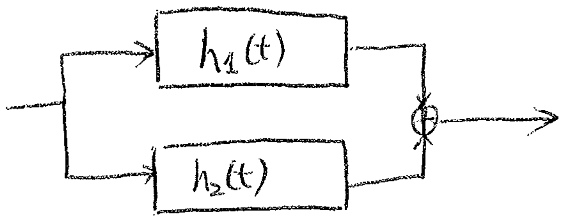
Signature:

Question 1: [35%, Work-out question] Consider two LTI systems with impulse responses being $h_1(t) = \delta(t-2)$ and

$$h_2(t) = \begin{cases} 1 & \text{if } -1 \le t < 0 \\ e^{-t} & \text{if } 0 \le t \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

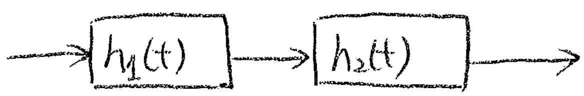
respectively. Answer the following questions.

1. [10%, Outcomes 2 and 3] Consider the following parallel combination of the two systems:

   Plot the overall impulse response $h_{\text{parallel}}(t)$ for the range $t = -2$ to 4.

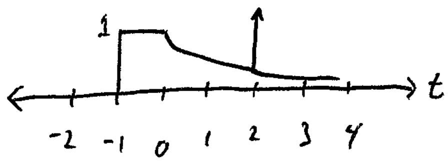
2. [10%, Outcomes 2 and 3] If I combine the two systems serially:

   what is the overall impulse response $h_{\text{serial}}(t)$? Plot $h_{\text{serial}}(t)$ for the range $t = -2$ to 4.

3. [15%, Outcomes 2 and 3] For an input $x(t) = e^t\mathcal{U}(-t)$, compute the output $y(t)$ of the serially concatenated system. If you do not know the answer for the serially concatenated joint system, you can compute the output $y_2(t)$ when passing through a single LTI system with impulse response $h_2(t)$. You can still get 12 pts if your answer is correct.

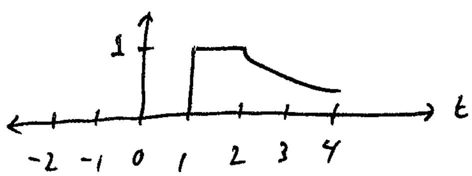
1. $h_{parallel}(t) = h_1(t) + h_2(t)$

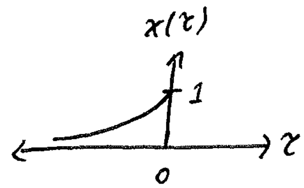


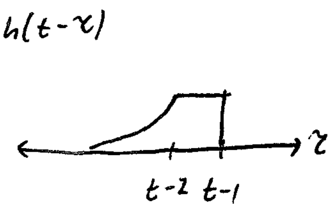
2. $h_{serial}(t) = h_1(t) * h_2(t)$

$= \delta(t-2) * h_2(t) = h(t-2)$

$= \begin{cases} 1 & 1 \le t < 2 \\ e^{-(t-2)} & t \ge 2 \\ 0 & ow \end{cases}$





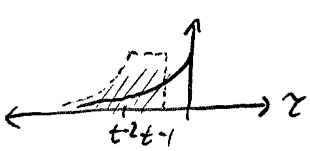


3. $x(t) = e^t u(-t)$

$y(t) = h_{serial}(t) * x(t)$

$= \int_{-\infty}^{\infty} x(\tau) h_{serial}(t-\tau) d\tau$

Case 1: $t-1<0, \ t<1$



$y(t) = \int_{-\infty}^{t-2} e^{\tau} e^{-(t-2)-\tau} d\tau + \int_{t-2}^{t-1} e^{\tau}(1) d\tau$

$= e^{-(t-2)} \int_{-\infty}^{t-2} e^{2\tau} d\tau + e^{\tau}\Big|_{t-2}^{t-1}$

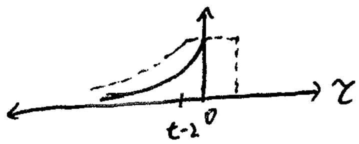
$= e^{-(t-2)} \frac{1}{2} e^{2\tau}\Big|_{-\infty}^{t-2} + (e^{t-1} - e^{t-2})$

$= \frac{1}{2} e^{-(t-2)} (e^{2(t-2)} - 1) + e^{t-1} - e^{t-2}$

$= \frac{1}{2}(e^{t-2} - e^{-(t-2)}) + e^{t-1} - e^{t-2}$

$= -\frac{1}{2} e^{t-2} - e^{-(t-2)} + e^{t-1}$

Case 2: $1 \le t < 2$



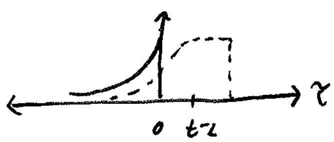
$$y(t) = \int_{-\infty}^{t-2} e^{\tau} e^{-(t-\tau-2)} d\tau + \int_{t-2}^{0} e^{\tau}(1)\, d\tau$$

$$= e^{-(t-2)} \int_{-\infty}^{t-2} e^{2\tau} d\tau + e^{\tau}\Big|_{t-2}^{0}$$

$$= \frac{1}{2}\left(e^{t-2} - e^{-(t-2)}\right) + 1 - e^{t-2}$$

$$= 1 - \frac{1}{2} e^{t-2} - e^{-(t-2)}$$

Case 3: $t \ge 2$



$$y(t) = \int_{-\infty}^{0} e^{\tau}\left(e^{-(t-\tau-2)}\right) d\tau$$

$$= e^{-(t-2)} \int_{-\infty}^{0} e^{2\tau} d\tau$$

$$= e^{-(t-2)} \frac{1}{2} e^{2\tau}\Big|_{-\infty}^{0} = \frac{1}{2} e^{-(t-2)}$$

$$y(t) = \begin{cases} -\frac{1}{2} e^{t-2} - e^{-(t-2)} + e^{t-1} & t < 1 \\ 1 - \frac{1}{2} e^{t-2} - e^{-(t-2)} & 1 \le t < 2 \\ \frac{1}{2} e^{-(t-2)} & t \ge 2 \end{cases}$$

Question 2: [20%, Work-out question, Outcome 4]

1. [10%] Consider a continuous-time signal

$$x(t) = \cos\left(\frac{3t+\pi}{2}\right) + \sin\left(5t + \frac{2\pi}{3}\right). \tag{2}$$

   Find its Fourier series representation

2. [10%] Consider a discrete-time signal

$$x[n] = \cos\left(\frac{2\pi n}{5}\right) + \sin\left(\frac{12\pi n}{5}\right). \tag{3}$$

   Find its Fourier series representation.

1. $T = 4\pi \qquad \omega_0 = \frac{2\pi}{4\pi} = \frac{1}{2}$

$$x(t) = \frac{1}{2}\left(e^{j\left(\frac{3}{2}t + \frac{\pi}{2}\right)} + e^{-j\left(\frac{3}{2}t + \frac{\pi}{2}\right)}\right) + \frac{1}{2j}\left(e^{j\left(5t + \frac{2\pi}{3}\right)} - e^{-j\left(5t + \frac{2\pi}{3}\right)}\right)$$

$$= \frac{1}{2}e^{j\frac{\pi}{2}}e^{j\frac{1}{2}3t} + \frac{1}{2}e^{-j\frac{\pi}{2}}e^{j\frac{1}{2}(-3t)} \qquad e^{j\frac{\pi}{2}} = j$$

$$+ \frac{1}{2j}e^{j\frac{2\pi}{3}}e^{j\frac{1}{2}(10)t} - \frac{1}{2j}e^{-j\frac{2\pi}{3}}e^{j\frac{1}{2}(-10)t}$$

$$a_k = \begin{cases} \frac{j}{2} & k = 3 \\ \frac{-j}{2} & k = -3 \\ \frac{1}{2j}e^{j\frac{2\pi}{3}} & k = 10 \\ \frac{-1}{2j}e^{-j\frac{2\pi}{3}} & k = -10 \\ 0 & \text{ow} \end{cases}$$

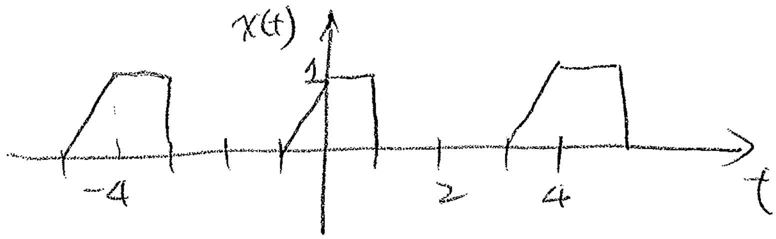
2. $N = 5 \qquad \omega_0 = \frac{2\pi}{5}$

$$x[n] = \frac{1}{2}\left(e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n}\right) + \frac{1}{2j}\left(e^{j\frac{12\pi}{5}n} - e^{-j\frac{12\pi}{5}n}\right)$$

$$= \frac{1}{2}\left(e^{j\frac{2\pi}{5}(1)n} + e^{j\frac{2\pi}{5}(-1)n}\right) + \frac{1}{2j}\left(e^{j\frac{2\pi}{5}(1)n} - e^{j\frac{2\pi}{5}(-1)n}\right)$$

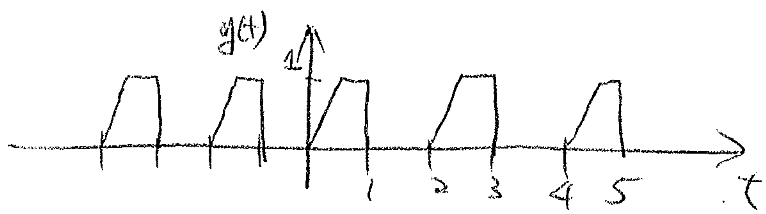
$$a_k = \begin{cases} \frac{1}{2} + \frac{1}{2j} & k = 1 \\ \frac{1}{2} - \frac{1}{2j} & k = -1 \\ 0 & \text{ow} \end{cases} = \begin{cases} \frac{1}{2} + \frac{1}{2j} & k = 1 \\ \frac{1}{2} - \frac{1}{2j} & k = 4 \\ 0 & \text{ow} \end{cases}$$

Question 3: [25%, Work-out question] Consider the following continuous time signal $x(t)$ with period 4:



Let $(a_k, \omega_x = \frac{2\pi}{4})$ denote the Fourier series representation of $x(t)$.

1. [5%, Outcome 4] What is the value of a_0?
2. [5%, Outcome 4] What is the value of $\sum_{k=-\infty}^{\infty} a_k$? Hint: Use the synthesis formula.
3. [5%, Outcome 4] What is the value of $\sum_{k=-\infty}^{\infty} a_k(-1)^k$? Hint: $(-1)^k = e^{jk\pi}$.
4. [5%, Outcome 4] What is the value of $\sum_{k=-\infty}^{\infty} |a_k|^2$?
5. [5%, Outcome 4] Consider another signal $y(t)$ that has period 2:



Let $(b_k, \omega_y = \frac{2\pi}{2})$ denote the Fourier series representation of $y(t)$. Write down the relationship between b_k and a_k.

1. $a_0 = \frac{1}{4}\int_{-2}^{2} x(t)\,dt = \frac{1}{4}\int_{-1}^{0} (t+1)\,dt + \frac{1}{4}\int_{0}^{1} (1)\,dt$

$= \frac{1}{4}\left[(\frac{1}{2}t^2 + t)\Big|_{-1}^{0} + t\Big|_0^1 \right] = \frac{1}{4}\left[\frac{1}{2}(0-(1)) + 0+1 + 1-0 \right]$

$= \frac{1}{4}(-\frac{1}{2}+2) = \frac{1}{4}(\frac{3}{2}) = \frac{3}{8}$

2. $x(0) = \sum_{k=-\infty}^{\infty} a_k = 1$

3. $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$t = 2 \Rightarrow \omega = \frac{2\pi}{2}k = \pi k$

$x(2) = \sum_{k=-\infty}^{\infty} a_k e^{j\pi k} = \sum_{k=-\infty}^{\infty} a_k (-1)^k = 0$

4. $\displaystyle\sum_{k=-\infty}^{\infty} |a_k|^2 = \frac{1}{T}\int_{-2}^{2} |x(t)|^2 \, dt$ by Parseval's Relation

$$= \frac{1}{4}\int_{-1}^{0} (t+1)^2 \, dt + \frac{1}{4}\int_{0}^{1} (1)^2 \, dt$$

$$= \frac{1}{4}\left[\int_{-1}^{0} (t^2+2t+1) \, dt + \int_{0}^{1} 1 \, dt \right]$$

$$= \frac{1}{4}\left[\left(\frac{1}{3}t^3 + t^2 + t\right)\Big|_{-1}^{0} + t\Big|_{0}^{1} \right]$$

$$= \frac{1}{4}\left[\frac{1}{3}(0+1) + (0-1) + (0+1) + (1-0) \right]$$

$$= \frac{1}{4}\left(\frac{1}{3} - 1 + 1 + 1 \right) = \frac{1}{4}\left(\frac{4}{3}\right)$$

$$= \frac{1}{3}$$

5. $y(t) = x(2t-1) =$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{\pi}{2}k(2t-1)} = \sum_{k=-\infty}^{\infty} \underbrace{a_k e^{-j\frac{\pi}{2}k}}_{b_k} e^{j\pi k t}$$

$$b_k = a_k e^{-j\frac{\pi}{2}k}$$

Question 4: [20%, Multiple Choices] Consider the following two systems, System 1 and System 2:

$$\text{System 1:} \quad y(t) = \int_{s=-\infty}^{\infty} x(s) 2^{t-s} \mathcal{U}(t-s) ds \tag{4}$$

$$\text{System 2:} \quad y[n] = \begin{cases} x[n-2] & \text{if } n \text{ is odd} \\ x[n/2] & \text{if } n \text{ is a multiple of 4} \\ 1 & \text{otherwise } (n \text{ is even but not a multiple of 4}) \end{cases} \tag{5}$$

1. [5%, Outcome 1] Are Systems 1 and 2 memoryless?
2. [5%, Outcome 1] Are Systems 1 and 2 causal?
3. [5%, Outcome 1] Are Systems 1 and 2 stable?
4. [5%, Outcome 1] Are Systems 1 and 2 linear?
5. [5%, Outcome 1] Are Systems 1 and 2 time-invariant?

System 1: $y(t) = x(t) * h(t)$ $\quad h(t) = 2^t \mathcal{U}(t)$

System 1	System 2
NOT memoryless	NOT memoryless
Causal	NOT causal
NOT stable	stable
linear	NOT linear
time-invariant	NOT time-invariant