ECE 301, Final exam of the session of Prof. Chih-Chun Wang
Friday 1pm–3pm , May 7, 2010, CL50 224.

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains both multiple-choice and work-out questions. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 24 pages in the exam booklet. Use the back of each page for rough work. The last pages are all the Tables. You may rip the last pages for easier reference. **Do not use your own copy of the Tables. Using your own copy of Tables will be considered as cheating.**
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

E-mail:

Signature:

Question 1: [20%] No need to write down justifications for this question.

1. [2%] $x(t) = \frac{(\cos(2\pi t))^2}{2+\sin(3\pi t)}$. Is $x(t)$ periodic?
2. [2%] $y(t) = \frac{\sin(2000\pi t)}{200t}$. Is $y(t)$ an odd signal?
3. [3%] Continue from the previous question. Is $y(t)$ of finite power? Is $y(t)$ of finite energy?
4. [3%] Consider an LTI system with impulse response $h(t)$. If we feed this system with an input $x(t) = e^{j2010t}\mathcal{U}(t+4) + \frac{1}{t^2+2}$, the output is $y(t) = e^{j(2010t-2010)}\mathcal{U}(t+3) + \frac{1}{(t-1)^2+2}$. Write down the expression of the impulse response $h(t)$.
5. [3%] Consider a system with the input/output relationship

$$y(t) = \int_{t^2-1}^{t^2} x(s)ds. \tag{1}$$

Is the system causal? Is the system linear? Is the system time-invariant?

6. [2%] $x[n] = e^{jn}$. Is $x[n]$ periodic?
7. [2%] $x[n] = \frac{\cos(1.5\pi n)}{n^4+1}$ and $X(e^{j\omega})$ is its Fourier transform. Is $X(e^{j\omega})$ periodic?
8. [3%] $x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{k^4+1}\delta(t - 0.3\pi k)$ and $X(j\omega)$ is its Fourier transform. Is $x[n]$ periodic? Is $X(j\omega)$ periodic?

1. ~~No~~ Yes $(\cos(2\pi t))^2$ has period $\frac{1}{2}$

$2+\sin(3\pi t)$ has period $\frac{2}{3}$

$x(t)$ has period 3

2. No. $y(-t) = \frac{\sin(-2000\pi t)}{200(-t)} = \frac{-\sin(2000\pi t)}{-200t} = y(t) \Rightarrow$ even

3. Finite 1.5 power & finite Energy 1.5

4. $h(t) = \delta(t-1)$

5. Non-causal / $y(-1) = \int_0^1 x(s)\,ds$

linear /

Not time invariant /

6. No. It is NOT periodic

7. $X(e^{j\omega})$ is always periodic.

8. $x(t)$ is NOT periodic: $\frac{1}{k^4+1}$ 1.5

1.5

$X(j\omega)$ is periodic

$$x(t) = y(t) \sum_{k=-\infty}^{\infty} \delta(t - 0.3\pi k)$$

$$y(t) = \frac{1}{(0.3\pi t)^4 + 1}$$

$$X(j\omega) = \frac{1}{2\pi} Y(j\omega) * \mathcal{F}\left\{ \sum_{k=-\infty}^{\infty} \delta(t - 0.3\pi k) \right\}$$

$$= \frac{1}{2\pi} Y(j\omega) * \frac{1}{0.3\pi} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{0.3\pi} k)$$

$$= \frac{1}{0.6\pi^2} \sum_{k=-\infty}^{\infty} Y(j(\omega - \frac{20}{3} k))$$

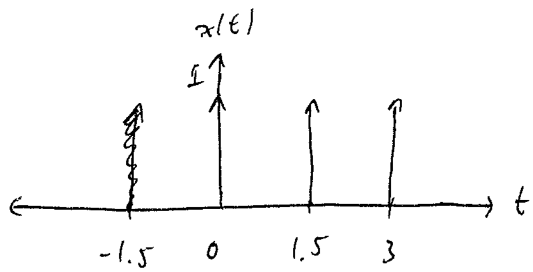
Question 2: [10%]

Consider $x(t) = \sum_{k=0}^{\infty} \delta(t - 1.5k)$, and

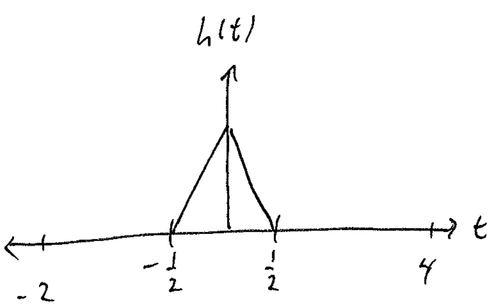
$$h(t) = \begin{cases} 2t+1 & \text{if } -0.5 < t < 0 \\ 1-2t & \text{if } 0 < t < 0.5 \\ 0 & \text{otherwise} \end{cases} . \qquad (2)$$

1. [2%] Draw $x(t)$ for the range $-2 < t < 4$.
2. [2%] Draw $h(t)$ for the range $-2 < t < 4$.
3. [6%] Draw $y(t) = x(t) * h(t)$ for the range $-2 < t < 4$.

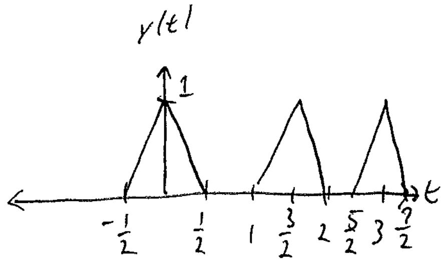
1.



2.



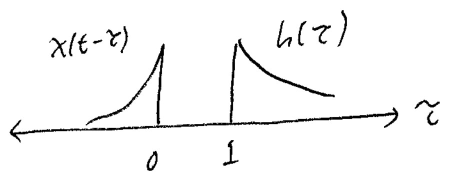
3. $y(t) = \sum_{k=0}^{\infty} h(t - 1.5k)$



Question 3: [10%] $x(t) = e^{-3t}\mathcal{U}(t)$ and $h(t) = e^{-3t}\mathcal{U}(t-1)$. Find the expression of $y(t) = x(t) * h(t)$.

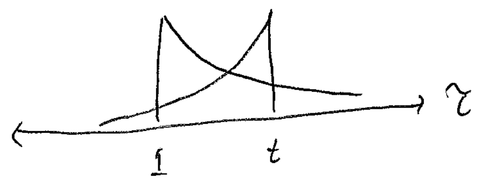
$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Region 1: $t < 1$



$$y(t) = 0$$

Region 2: $t > 1$



$$y(t) = \int_1^t e^{-3(t-\tau)}\, e^{-3\tau}\, d\tau$$

$$= e^{-3t} \int_1^t d\tau$$

$$= e^{-3t}(t-1)$$

$$y(t) = \begin{cases} e^{-3t}(t-1) & t > 1 \\ 0 & t < 1 \end{cases}$$

$$y(t) = (t-1)\, e^{-3t}\, \mathcal{U}(t-1)$$

Question 4: [15%] Consider the following difference equation:

$$y[n] = y[n-1] - \frac{2}{9}y[n-2] + 3x[n] - \frac{5}{3}x[n-1]. \tag{3}$$

1. [5%] Find the frequency response $H(e^{j\omega})$.
2. [5%] When the input is $x[n] = \delta[n]$, find out the output $y[n]$. If you do not know the answer $H(e^{j\omega})$ of the previous question, you can assume that

$$H(e^{j\omega}) = \begin{cases} 1 & \text{if } |\omega| < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le |\omega| < \pi \end{cases}. \tag{4}$$

3. [5%] When the input is $x[n] = e^{jn}$, find out the output $y[n]$. If you do not know the answers to the previous questions, you can assume that

$$H(e^{j\omega}) = \begin{cases} 1 & \text{if } |\omega| < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le |\omega| < \pi \end{cases}. \tag{5}$$

1. $Y(e^{j\omega}) = e^{-j\omega}Y(e^{j\omega}) - \frac{2}{9}e^{-j2\omega}Y(e^{j\omega}) + 3X(e^{j\omega}) - \frac{5}{3}e^{-j\omega}X(e^{j\omega})$

$Y(e^{j\omega})\left(1 - e^{-j\omega} + \frac{2}{9}e^{-j2\omega}\right) = X(e^{j\omega})\left(3 - \frac{5}{3}e^{-j\omega}\right)$

$H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{3 - \frac{5}{3}e^{-j\omega}}{1 - e^{-j\omega} + \frac{2}{9}e^{-j2\omega}}$

2. $X(e^{j\omega}) = 1$

$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = H(e^{j\omega}) \Rightarrow y[n] = h[n]$

$$H(e^{j\omega}) = \frac{3 - \frac{5}{3}e^{-j\omega}}{1 - e^{-j\omega} + \frac{2}{9}e^{-j2\omega}} = \cancel{\frac{3 - \frac{5}{3}e^{-j\omega}}{(e^{-j\omega} - \frac{3}{2})(e^{-j\omega} - 3)}}$$

$$e^{-j\omega} = \frac{1 \pm \sqrt{1 - 4(\frac{2}{9})(1)}}{2(\frac{2}{9})} = \frac{1 \pm \sqrt{\frac{1}{9}}}{\frac{4}{9}} = (1 \pm \tfrac{1}{3})\tfrac{9}{4} = \tfrac{9}{4} \pm \tfrac{3}{4} = \tfrac{3}{2}, 3$$

$$= \frac{3 - \frac{5}{3}e^{-j\omega}}{(\frac{2}{3}e^{-j\omega} - 1)(\frac{1}{3}e^{-j\omega} - 1)} = \frac{A}{1 - \frac{2}{3}e^{-j\omega}} + \frac{B}{1 - \frac{1}{3}e^{-j\omega}}$$

$$A(1 - \tfrac{1}{3}e^{-j\omega}) + B(1 - \tfrac{2}{3}e^{-j\omega}) = 3 - \tfrac{5}{3}e^{-j\omega}$$

$A + B = 3$ $\qquad$ $B = 3 - A$

$-\frac{1}{3}A - \frac{2}{3}B = -\frac{5}{3}$ $\qquad$ $A + 2B = 5$ $\qquad$ $A + 6 - 2A = 5$

$-A = -1$

$A = 1$ $\qquad$ $B = 2$

$$H(e^{j\omega}) = \frac{1}{1 - \frac{2}{3}e^{-j\omega}} + \frac{2}{1 - \frac{1}{3}e^{-j\omega}}$$

$$h[n] = \left(\tfrac{2}{3}\right)^n u[n] + 2\left(\tfrac{1}{3}\right)^n u[n]$$

3. $y[n] = H(e^{j(1)})\, e^{jn}$

$$= \frac{3 - \frac{5}{3}e^{-j}}{1 - e^{-j} + \frac{2}{9}e^{-j2}}\, e^{jn}$$

Question 5: [15%]

Prof. Wang wanted to transmit an AM-SSB signal. To that end, he wrote the following MATLAB code.

```
% Initialialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read two different .wav files
[x1, f_sample, N]=wavread('x1');
x1=x1';
[x2, f_sample, N]=wavread('x2');
x2=x2';

% Step 1: Make the signals band-limited.
W_M=2000*pi;
h=1/(pi*t).*(sin(W_M*t));
x1_new=ece301conv(x1, h);
x2_new=ece301conv(x2, h);

% Step 2: Multiply x_new with a cosine wave.
x1_h=x1_new.*cos(5000*pi*t);
x2_h=x2_new.*cos(7000*pi*t);
h1=1/(pi*t).*(sin(5000*pi*t));
h2=1/(pi*t).*(sin(7000*pi*t));

% Step 3: Keep one of the side bands
x1_sb=x1_h-ece301conv(x1_h, h1);
x2_sb=x2_h-ece301conv(x2_h, h2);

% Step 4: create the transmitted signal
y=x1_sb+x2_sb;
wavwrite(y', f_sample, N, 'y.wav');
```

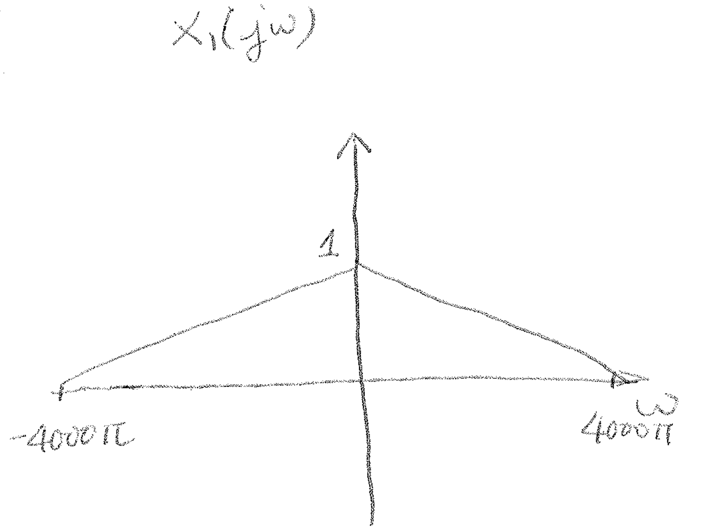
$2\pi(1000)t \Rightarrow f_c = 1\,kHz$

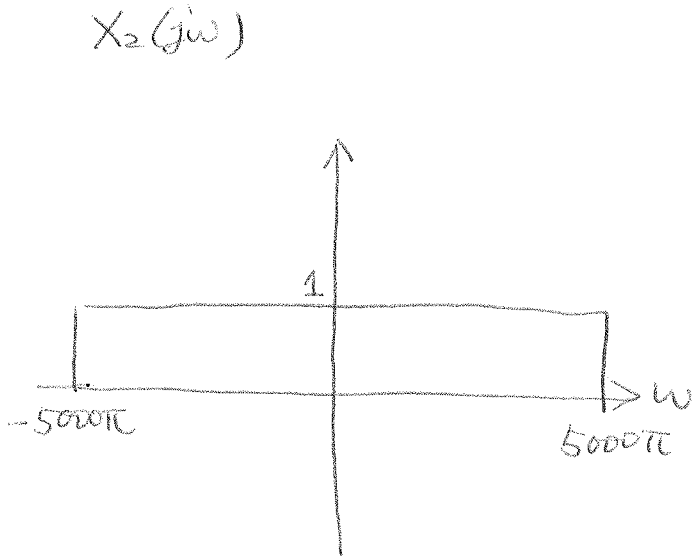
centers @ 2.5 kHz & 3.5 kHz

1. [2%] Is this system using the upper or the lower side band?
2. [6%] The frequency spectrums of x1 and x2 are described in the following figures.

1. Upper side band

2.





Plot the frequency spectrum of x1_h and y.

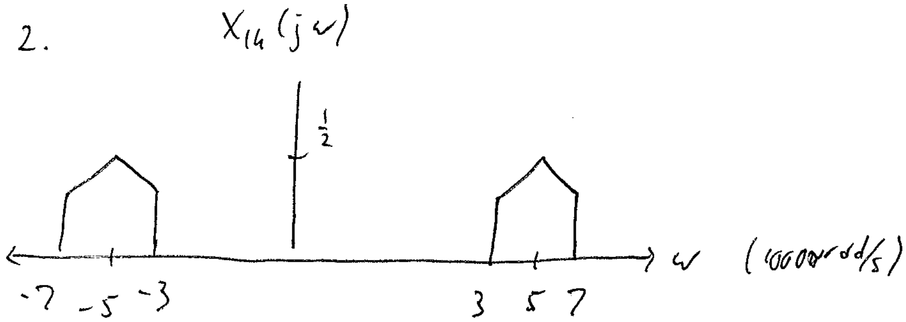
Knowing that Prof. Wang used the above code to generate the "y.wav" file, a student tried to demodulate the output waveform "y.wav" by the following code.

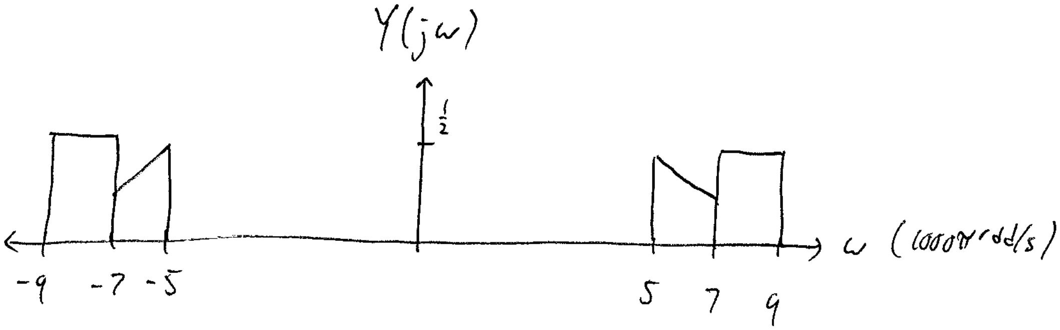
```
% Initialialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read the .wav files
[y, f_sample, N]=wavread('y');
y=y';

% Create the low-pass filter.
h_M=1/(pi*t).*(sin(2000*pi*t));

% demodulate signal 1
y1=4*y.*cos(5000*pi*t);
x1_hat=ece301conv(y1,h_M);

wavplay(x1_hat,f_sample)

% demodulate signal 2
y2=4*y.*cos(7000*pi*t);
x2_hat=ece301conv(y2,h_M);

wavplay(x2_hat,f_sample)
```

3. [3%] Can the student demodulate x2 successfully without noise (also known as interference)? Use one or two sentences to briefly explain your answer.

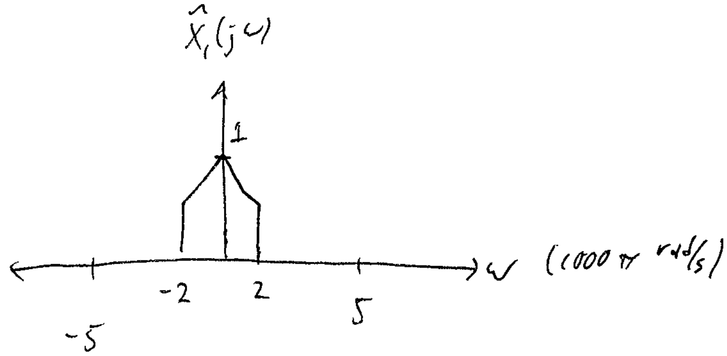
4. [4%] Can the student demodulate x1 successfully without noise (also known as interference)? Use one or two sentences to briefly explain your answer.

2.





3.

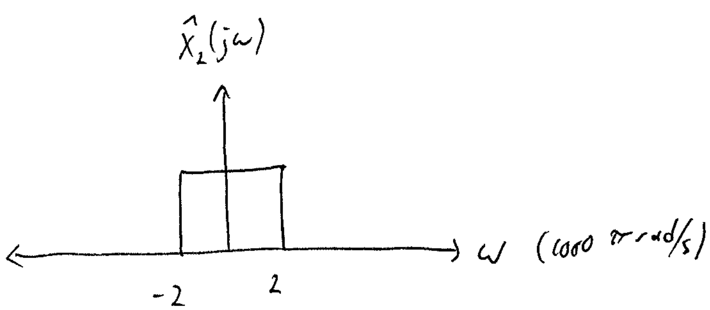


Yes :

or ~~No. The original BPF cut out all frequencies above 2000π rad/s (1kHz).~~

No interference in this case.

4.



No. The original BPF cut out all frequencies above 2000π rad/s (1kHz).

due to interference

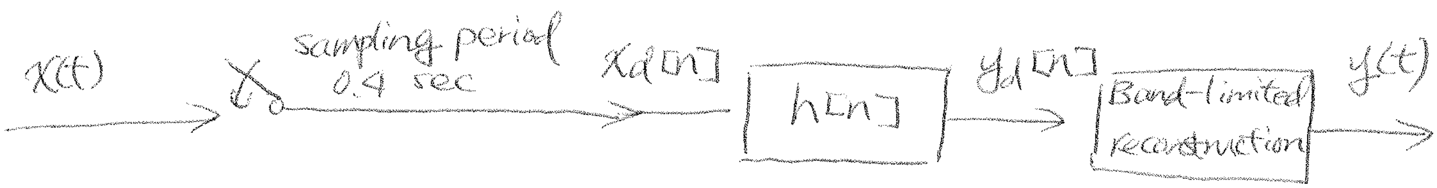
Question 6: [15%]

1. [1%] Given a signal $x(t)$, write down the equation how to convert $x(t)$ into its sample values $x_d[n]$ when the sampling period is 0.4 sec.

2. [2%] Suppose we know that $x_d[n] = \delta[n-2]$. We use linear interpolation to reconstruct the original signal, and denote the reconstructed output as $x_1(t)$. Plot $x_1(t)$ for the range $-1 < t < 2$.

3. [4%] If we use a band-limited interpolation to reconstruct the original signal, and denote the reconstructed output as $x_2(t)$, write down the expression of $x_2(t)$ and plot $x_2(t)$ for the range $-1 < t < 2$.

4. [8%] Consider the following digital signal processing system.



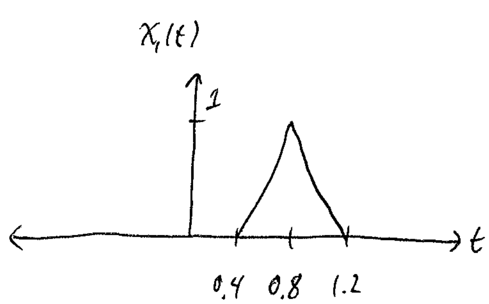
Suppose the $h[n]$ has its DTFT being

$$H(e^{j\omega}) = \begin{cases} 1 & \text{if } |\omega| < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq |\omega| < \pi \end{cases}. \tag{6}$$

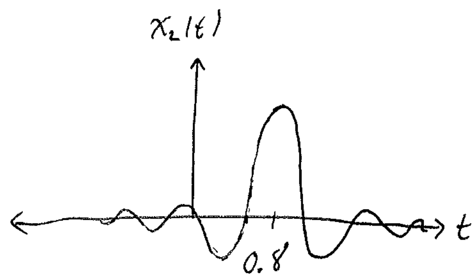
When the input $x(t) = \cos(\pi t) + \sin(0.5\pi t)$, what is the output $y(t)$?

1. $x_d[n] = x(n\,0.4)$

2.



3.



$$\frac{\sin\left(\frac{\pi}{0.4}(t-0.8)\right)}{\frac{\pi}{0.4}\cdot(t-0.8)}$$

4. $x(t) = \cos(\pi t) + \sin(\frac{\pi}{2} t)$

$x_d[n] = \cos(\frac{2}{5}\pi n) + \sin(\frac{\pi}{5} n)$

$$X_d(e^{j\omega}) = \frac{1}{2}\delta(\omega - \frac{2}{5}\pi) + \frac{1}{2}\delta(\omega + \frac{2}{5}\pi)$$
$$+ \frac{1}{2j}\delta(\omega - \frac{\pi}{5}) - \frac{1}{2j}\delta(\omega + \frac{\pi}{5}) \quad \text{for} \quad |\omega| < \pi$$

$\omega_s = \frac{2\pi}{2/5} = 5\pi$ rad(s)

$2(\pi) < 2\omega_s$ $\Rightarrow$ perfect reconstruction

$y(t) = x(t)$

Question 7: [15%]

1. [3%] $x[n] = \left(-\frac{1}{2}\right)^n \mathcal{U}[-n]$. Find the corresponding Z-transform $X(z)$ and plot the corresponding ROC, zeros, and poles.

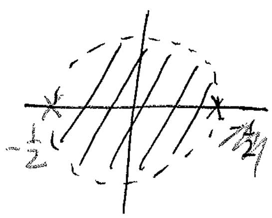
2. [2%] $X(z) = 2z^{-2} + 10z^3$ and the corresponding ROC is the entire Z-plane except for $z = 0$ and $z = \infty$. Find the corresponding $x[n]$.

3. [5%] We know that $X(z) = \frac{1}{(1-\frac{2}{3}z^{-1})(1-4z^{-1})}$ and the Fourier transform of the corresponding $x[n]$ exists. Find $x[n]$.

4. [5%] Suppose $x[n] = 2^n \mathcal{U}[-n-1]$ and $h[n] = (0.25)^n \mathcal{U}[-n-1]$. Let $y[n] = x[n] * h[n]$. Find the Z-transform $Y(z)$.

1. $X(z) = \sum_{n=-\infty}^{0} \left(-\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{-n} z^{n} = \sum_{n=0}^{\infty} (-2z)^n$

$= \frac{1}{1+2z}$ $\quad |-2z| < 1 \quad \Rightarrow \quad |z| < \frac{1}{2}$



2. $x[n] = 2\delta[n-2] + 10\delta[n+3]$

3.

$X(z) = \frac{A}{1-\frac{2}{3}z^{-1}} + \frac{B}{1-4z^{-1}}$ $\quad A(1-4z^{-1}) + B(1-\frac{2}{3}z^{-1}) = 1$

$A + B = 1$

$-4A - \frac{2}{3}B = 0$ $\quad B = -6A$ $\quad A - 6A = 1 \Rightarrow A = -\frac{1}{5}$, $B = \frac{6}{5}$

$X(z) = \frac{-1/5}{1-\frac{2}{3}z^{-1}} + \frac{6/5}{1-4z^{-1}}$ $\quad$ ROC: $\frac{2}{3} < |z| \cap |z| < 4$

$x[n] = -\frac{1}{5}\left(+\frac{2}{3}\right)^n \mathcal{U}[n] - \frac{6}{5}(4)^n \mathcal{U}[-n-1]$

4. $x[n] = 2^n u[-n-1]$ $\qquad h[n] = \left(\frac{1}{4}\right)^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{-1} 2^n z^{-n} = \sum_{k=0}^{\infty} 2^{-k-1} z^{k+1} = \frac{1}{2} z \sum_{k=0}^{\infty} \left(\frac{1}{2} z\right)^k = \frac{1}{2} z \frac{1}{1-\frac{1}{2}z}$$

$k = -n-1 \qquad n = -k-1$

$$= \frac{-1}{1-\frac{1}{2}z^{-1}} \qquad \text{ROC: } |z| < 2$$

$$H(z) = \frac{-1}{1-4z^{-1}} \qquad \text{ROC: } |z| < \frac{1}{4}$$

$$Y(z) = X(z)\,H(z)$$

$$= \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-4z^{-1}\right)} \qquad \text{RoC: } |z| < \frac{1}{4}$$