Midterm #3 of ECE301, Prof. Wang's section

8–9pm Wednesday, November 17, 2010, WTHR 172.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

E-mail:

Signature:

Question 1: [25%, Work-out question]

1. [10%, Outcome 4] Consider the following DT signal:

$$x_1[n] = \begin{cases} (e^j)^n & \text{if } 0 \le n \le 100 \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

Find the expression of $X_1(e^{j\omega})$, the Fourier transform of $x_1[n]$.

2. [15%, Outcome 4] Consider the following DT signal:

$$x_2[n] = j\sin\left(\frac{100\pi}{3}n\right). \tag{2}$$

Find the expression of $X_2(e^{j\omega})$, the Fourier transform of $x_2[n]$. Plot $X_2(e^{j\omega})$ for the range $-\pi < \omega < \pi$.

1. $$X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1[n] e^{-j\omega n} = \sum_{n=0}^{100} e^{jn} e^{-j\omega n} = \sum_{n=0}^{100} \left(e^{j(1-\omega)}\right)^n$$

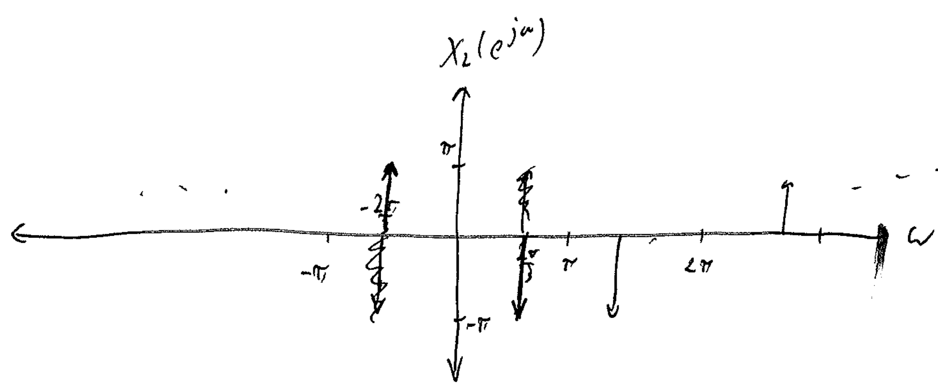
$$= \frac{1 - e^{j(1-\omega)101}}{1 - e^{j(1-\omega)}} = \frac{e^{+j(1-\omega)\frac{101}{2}}\left(e^{-j(1-\omega)\frac{101}{2}} - e^{j(1-\omega)\frac{101}{2}}\right)}{e^{j\frac{1}{2}(1-\omega)}\left(e^{-j\frac{1}{2}(1-\omega)} - e^{j\frac{1}{2}(1-\omega)}\right)}$$

$$= e^{j(1-\omega)\frac{100}{2}} \frac{-2j\sin\left(\frac{101}{2}(1-\omega)\right)}{-2j\sin\left(\frac{1}{2}(1-\omega)\right)} = e^{j50(1-\omega)} \frac{\sin\left(\frac{101}{2}(1-\omega)\right)}{\sin\left(\frac{1}{2}(1-\omega)\right)}$$

2. $$x_2[n] = j\sin\left(\frac{100\pi}{3}n\right) = j\sin\left(\frac{102\pi}{3}n - \frac{2\pi}{3}n\right) = j\sin\left(34\pi n - \frac{2\pi}{3}n\right) = j\sin\left(-\frac{2\pi}{3}n\right) = -j\sin\left(\frac{2\pi}{3}n\right)$$

$$X_2(e^{j\omega}) = -j\frac{\pi}{j}\sum_{k=-\infty}^{\infty}\left[\delta\left(\omega - \frac{2\pi}{3} - 2\pi k\right) - \delta\left(\omega + \frac{2\pi}{3} - 2\pi k\right)\right]$$

$$= \pi \sum_{k=-\infty}^{\infty}\left[-\delta\left(\omega - \frac{2\pi}{3} - 2\pi k\right) + \delta\left(\omega + \frac{2\pi}{3} - 2\pi k\right)\right]$$



Question 2: [25%, Work-out question] Consider a DT LTI system, for which the input/output relationship is described by:

$$6y[n] + 5y[n-1] + y[n-2] = 7x[n] + 3x[n-1]. \qquad (3)$$

1. [10%, Outcomes 2, 5] Find out the impulse response $h[n]$ of this system.
2. [15%, Outcomes 2, 5] If the input is $x[n] = \left(-\frac{1}{3}\right)^n \mathcal{U}[n]$, find out the output $y[n]$. Hint: You may need to use the tables.

   If you do not know the answer to the previous question, you can assume $h[n] = \delta[n] + \left(-\frac{1}{3}\right)^n \mathcal{U}[n] + (1/4)^n \mathcal{U}[n]$. You will still receive full credit if your answer is correct.

1. $6Y(e^{j\omega}) + 5e^{-j\omega}Y(e^{j\omega}) + e^{-j2\omega}Y(e^{j\omega}) = 7X(e^{j\omega}) + 3e^{-j\omega}X(e^{j\omega})$

$Y(e^{j\omega})(6 + 5e^{-j\omega} + e^{-j2\omega}) = X(e^{j\omega})(7 + 3e^{-j\omega})$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{7+3e^{-j\omega}}{(3+e^{-j\omega})(2+e^{-j\omega})} = \frac{A}{3+e^{-j\omega}} + \frac{B}{2+e^{-j\omega}}$$

$A(2+e^{-j\omega}) + B(3+e^{-j\omega}) = 7 + 3e^{-j\omega}$

$2A + 3B = 7$ $\quad$ $B = 3 - A$ $\quad$ $-A = -2$

$A + B = 3$ $\quad$ $2A + 3(3-A) = 7$ $\quad$ $A = 2$

$2A + 9 - 3A = 7$ $\quad$ $B = 1$

$$H(e^{j\omega}) = \frac{2}{3+e^{-j\omega}} + \frac{1}{2+e^{-j\omega}}$$

Note: $a^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$

$$= \frac{2}{3}\frac{1}{1+\frac{1}{3}e^{-j\omega}} + \frac{1}{2}\frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

$h[n] = \frac{2}{3}\left(\frac{-1}{3}\right)^n u[n] + \frac{1}{2}\left(\frac{-1}{2}\right)^n u[n]$

2. $X(e^{j\omega}) = \frac{1}{1+\frac{1}{3}e^{-j\omega}}$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{3(7+3e^{-j\omega})}{(2+e^{-j\omega})(3+e^{-j\omega})^2} = \frac{A}{2+e^{-j\omega}} + \frac{B}{3+e^{-j\omega}} + \frac{C}{(3+e^{-j\omega})^2}$$

$A(9 + 6e^{-j\omega} + e^{-j2\omega}) + B(6 + 5e^{-j\omega} + e^{-j2\omega}) + C(2 + e^{-j\omega}) = 21 + 9e^{-j\omega}$

$9A + 6B + 2C = 21$ $\quad$ $B = -A$ $\quad$ $C = 9 - A$ $\quad$ $C = 6$

$6A + 5B + C = 9$ $\quad$ $3A + 2C = 21$ $\quad$ $3A + 2(9-A) = 21$ $\quad$ $B = -3$

$A + B = 0$ $\quad$ $A + C = 9$ $\quad$ $3A + 18 - 2A = 21$

$A = 3$

$$Y(e^{j\omega}) = \frac{3}{2+e^{-j\omega}} + \frac{-3}{3+e^{-j\omega}} + \frac{6}{(3+e^{-j\omega})^2}$$

$$= \frac{3}{2}\frac{1}{1+2e^{-j\omega}} - \frac{1}{1+\frac{1}{3}e^{-j\omega}} + \frac{2}{3}\frac{1}{(1+\frac{1}{3}e^{-j\omega})^2}$$

$$y[n] = \frac{3}{2}\left(-\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[n] + \frac{2}{3}(n+1)\left(-\frac{1}{3}\right)^n u[n]$$

Alternate Solution For $h[n] = \delta[n] + \left(-\frac{1}{3}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n]$

$$H(e^{j\omega}) = 1 + \frac{1}{1+\frac{1}{3}e^{-j\omega}} + \frac{1}{1-\frac{1}{4}e^{-j\omega}} = 1 + \frac{3}{3+e^{-j\omega}} + \frac{4}{4-e^{-j\omega}}$$

$$= \frac{(3+e^{-j\omega})(4-e^{-j\omega}) + 3(4-e^{-j\omega}) + 4(3+e^{-j\omega})}{(3+e^{-j\omega})(4-e^{-j\omega})} =$$

$$= \frac{12 + e^{-j\omega} - e^{-j2\omega} + 12 - 3e^{-j\omega} + 12 + 4e^{-j\omega}}{(3+e^{-j\omega})(4-e^{-j\omega})} = \frac{36 + 2e^{-j\omega} - e^{-j2\omega}}{(3+e^{-j\omega})(4-e^{-j\omega})}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{3(36 + 2e^{-j\omega} - e^{-j2\omega})}{(4-e^{-j\omega})(3+e^{-j\omega})^2} = \frac{A}{4-e^{-j\omega}} + \frac{B}{3+e^{-j\omega}} + \frac{C}{(3+e^{-j\omega})^2}$$

$$A(9 + 6e^{-j\omega} + e^{-j2\omega}) + B(12 + e^{-j\omega} - e^{-j2\omega}) + C(4 - e^{-j\omega}) = 108 + 6e^{-j\omega} - 3e^{-j2\omega}$$

$9A + 12B + 4C = 108$ $\qquad B = A + 3$ $\qquad C = 7A - 3$

$6A + B - C = 6$ $\qquad 21A + 4C = 72$ $\qquad 21A + 28A - 12 = 72$

$A - B = -3$ $\qquad 7A - C = 3$ $\qquad 49A = 84$

$$A = \frac{84}{49} \qquad C = \frac{84}{7} - 3 = \frac{84-21}{7} = 9 \qquad B = \frac{84+147}{49} = \frac{231}{49}$$

$$Y(e^{j\omega}) = \frac{84}{49}\frac{1}{4-e^{-j\omega}} + \frac{231}{49}\frac{1}{3+e^{-j\omega}} + 9\frac{1}{(3+e^{-j\omega})^2}$$

$$= \frac{21}{49}\frac{1}{1-\frac{1}{4}e^{-j\omega}} + \frac{77}{49}\frac{1}{1+\frac{1}{3}e^{-j\omega}} + \frac{1}{(1+\frac{1}{3}e^{-j\omega})^2}$$

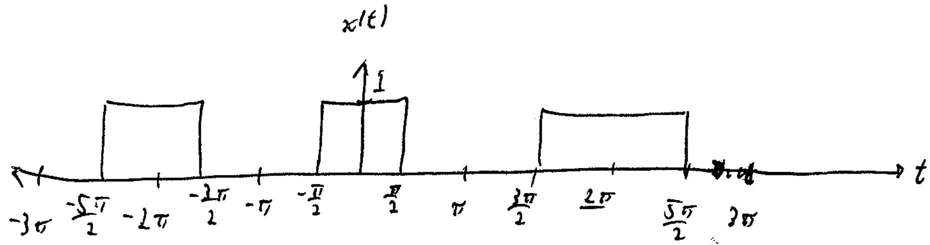
$$y[n] = \frac{21}{49}\left(\frac{1}{4}\right)^n u[n] + \frac{77}{49}\left(-\frac{1}{3}\right)^n u[n] + (n+1)\left(-\frac{1}{3}\right)^n u[n]$$

Question 3: [20%, Work-out question] Consider a periodic signal $x(t)$ of period 2π and

$$x(t) = \begin{cases} 1 & \text{if } |t| < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le |t| < \pi \end{cases} \tag{4}$$

1. [5%, Outcome 1] Plot $x(t)$ for the range $-3\pi < t < 3\pi$.
2. [15%, Outcomes 4, 6] Find the expression of $X(j\omega)$ and plot it for the range $-3.5 < \omega < 3.5$.

1.



2. First, find the Fourier Series Coefficients, a_k

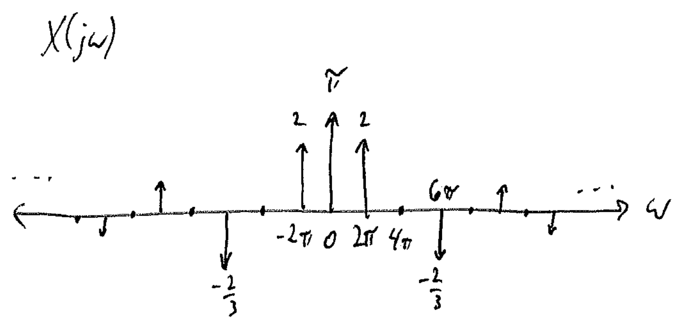
$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-j\frac{2\pi}{2\pi}kt}\,dt = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-jkt}\,dt = \frac{1}{2\pi}\frac{-1}{jk}e^{-jkt}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi k}\frac{-1}{j2}\left(e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k}\right) = \frac{\sin(\frac{\pi}{2}k)}{\pi k}$$

$$X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\,\delta\left(\omega - \frac{2\pi}{2\pi}k\right)$$

$$= 2\pi\sum_{k=-\infty}^{\infty}\frac{\sin(\frac{\pi}{2}k)}{\pi k}\,\delta(\omega - 2\pi k)$$

$$= \sum_{k=-\infty}^{\infty}\frac{2\sin(\frac{\pi}{2}k)}{k}\,\delta(\omega - 2\pi k)$$

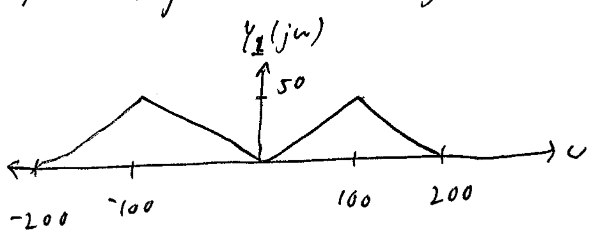


Question 4: [30%, Work-out question] Consider the following signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} \omega + 100 & \text{if } -100 < \omega < 0 \\ 100 - \omega & \text{if } 0 < \omega < 100 \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

1. [12%, Outcomes 4, 5] $y_1(t) = x(t)\cos(100t)$. Plot $Y_1(j\omega)$ for the range of $-200 < \omega < 200$.
2. [3%, Outcome 4] Compute the value of $\int_{-\infty}^{\infty} y_1(t)dt$.
3. [3%, Outcome 4] Compute the value of $\int_{-\infty}^{\infty} |y_1(t)|^2 dt$.
4. [12%, Outcomes 4, 5] $y_2(t) = x(t) \cdot j\sin(50t)$. Plot $Y_2(j\omega)$ for the range of $-200 < \omega < 200$.

1. $Y_1(j\omega) = \frac{1}{2}X(j(\omega-100)) + \frac{1}{2}X(j(\omega+100))$



2. $Y_1(j\omega) = \int_{-\infty}^{\infty} y_1(t) e^{-j\omega t} dt$

$Y_1(j0) = \int_{-\infty}^{\infty} y_1(t)\, dt = 0$

3. $\int_{-\infty}^{\infty} |y_1(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y_1(e^{j\omega})|^2 d\omega =$

$= \frac{1}{2\pi}(2)\left[\int_{-100}^{0} \frac{1}{4}(\omega+100)^2 d\omega + \int_{0}^{100} \frac{1}{4}(100-\omega)^2 d\omega\right] = \frac{1}{4\pi}\left[\int_{-100}^{0} (\omega^2 + 200\omega + 10^4) d\omega + \int_0^{100} (10^4 - 200\omega + \omega^2) d\omega\right]$

$= \frac{1}{4\pi}\left[\left(\frac{1}{3}\omega^3 + 100\omega^2 + 10^4\omega\right)\Big|_{-100}^{0} + \left(10^4\omega - 100\omega^2 + \frac{1}{3}\omega^3\right)\Big|_0^{100}\right]$

$= \frac{1}{4\pi}\left[\left(-\frac{-10^6}{3} + 10^6 - 10^6\right) + \left(10^6 - 10^6 + \frac{10^6}{3}\right)\right] = \frac{1}{4\pi}\left(\frac{10^6}{3} - 10^6 + 10^6 + 10^6 - 10^6 + \frac{10^6}{3}\right)$

$= \frac{1}{4\pi}\left(\left(\frac{2}{3}+2\right)10^6 - 2(10^6)\right) = \frac{1}{4\pi}\left(\frac{8}{3}(10^6) - 2(10^6)\right)$

$= \frac{\frac{2}{3}(10^6)}{4\pi} = \frac{10^6}{2\pi}$

4. $y_2(t) = j\sin(50t)\,x(t)$

$$Y_2(j\omega) = \frac{1}{2\pi} X(j\omega) * \pi[\delta(\omega-50) - \delta(\omega+50)]$$

$$= \frac{1}{2} X(j(\omega-50)) - \frac{1}{2} X(j(\omega+50))$$