**Midterm #3 of ECE301, Prof. Wang's section**
8–9pm Wednesday, November 17, 2010, WTHR 172.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**

2. This is a closed book exam.

3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.

4. Neither calculators nor help sheets are allowed.

Name:

Student ID:

E-mail:

Signature:

*Question 1:* [25%, Work-out question]

1. [10%, Outcome 4] Consider the following DT signal:

$$x_1[n] = \begin{cases} (e^j)^n & \text{if } 0 \le n \le 100 \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

   Find the expression of $X_1(e^{j\omega})$, the Fourier transform of $x_1[n]$.

2. [15%, Outcome 4] Consider the following DT signal:

$$x_2[n] = j \sin\left(\frac{100\pi}{3} n\right). \tag{2}$$

   Find the expression of $X_2(e^{j\omega})$, the Fourier transform of $x_2[n]$. Plot $X_2(e^{j\omega})$ for the range $-\pi < \omega < \pi$.

*Question 2:* [25%, Work-out question] Consider a DT LTI system, for which the input/output relationship is described by:

$$6y[n] + 5y[n-1] + y[n-2] = 7x[n] + 3x[n-1]. \tag{3}$$

1. [10%, Outcomes 2, 5] Find out the impulse response $h[n]$ of this system.

2. [15%, Outcomes 2, 5] If the input is $x[n] = \left(-\frac{1}{3}\right)^n \mathcal{U}[n]$, find out the output $y[n]$. Hint: You may need to use the tables.

   If you do not know the answer to the previous question, you can assume $h[n] = \delta[n] + \left(-\frac{1}{3}\right)^n \mathcal{U}[n] + (1/4)^n \mathcal{U}[n]$. You will still receive full credit if your answer is correct.

*Question 3:* [20%, Work-out question] Consider a periodic signal $x(t)$ of period $2\pi$ and

$$x(t) = \begin{cases} 1 & \text{if } |t| < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq |t| < \pi \end{cases} \tag{4}$$

1. [5%, Outcome 1] Plot $x(t)$ for the range $-3\pi < t < 3\pi$.
2. [15%, Outcomes 4, 6] Find the expression of $X(j\omega)$ and plot it for the range $-3.5 < \omega < 3.5$.

*Question 4:* [30%, Work-out question] Consider the following signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} \omega + 100 & \text{if } -100 < \omega < 0 \\ 100 - \omega & \text{if } 0 < \omega < 100 \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

1. [12%, Outcomes 4, 5] $y_1(t) = x(t)\cos(100t)$. Plot $Y_1(j\omega)$ for the range of $-200 < \omega < 200$.
2. [3%, Outcome 4] Compute the value of $\int_{-\infty}^{\infty} y_1(t)dt$.
3. [3%, Outcome 4] Compute the value of $\int_{-\infty}^{\infty} |y_1(t)|^2 dt$.
4. [12%, Outcomes 4, 5] $y_2(t) = x(t) \cdot j\sin(50t)$. Plot $Y_2(j\omega)$ for the range of $-200 < \omega < 200$.

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(j\omega) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

| Property | Section | Periodic Signal | Fourier Series Coefficients |
|---|---|---|---|
| | | $\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$ | $a_k$ <br> $b_k$ |
| Linearity | 3.5.1 | $Ax(t) + By(t)$ | $Aa_k + Bb_k$ |
| Time Shifting | 3.5.2 | $x(t - t_0)$ | $a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$ |
| Frequency Shifting | | $e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$ | $a_{k-M}$ |
| Conjugation | 3.5.6 | $x^*(t)$ | $a^*_{-k}$ |
| Time Reversal | 3.5.3 | $x(-t)$ | $a_{-k}$ |
| Time Scaling | 3.5.4 | $x(\alpha t), \alpha > 0$ (periodic with period $T/\alpha$) | $a_k$ |
| Periodic Convolution | | $\int_T x(\tau)y(t-\tau)d\tau$ | $Ta_k b_k$ |
| Multiplication | 3.5.5 | $x(t)y(t)$ | $\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$ |
| Differentiation | | $\frac{dx(t)}{dt}$ | $jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$ |
| Integration | | $\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$) | $\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$ |
| Conjugate Symmetry for Real Signals | 3.5.6 | $x(t)$ real | $\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | 3.5.6 | $x(t)$ real and even | $a_k$ real and even |
| Real and Odd Signals | 3.5.6 | $x(t)$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | | $\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$ | $\Re e\{a_k\}$ <br> $j\Im m\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

## Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

## 3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

**TABLE 3.2** PROPERTIES OF DISCRETE-TIME FOURIER SERIES

| Property | Periodic Signal | Fourier Series Coefficients |
|---|---|---|
| | $\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period $N$ and fundamental frequency $\omega_0 = 2\pi/N$ | $\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period $N$ |
| Linearity | $Ax[n] + By[n]$ | $Aa_k + Bb_k$ |
| Time Shifting | $x[n - n_0]$ | $a_k e^{-jk(2\pi/N)n_0}$ |
| Frequency Shifting | $e^{jM(2\pi/N)n} x[n]$ | $a_{k-M}$ |
| Conjugation | $x^*[n]$ | $a^*_{-k}$ |
| Time Reversal | $x[-n]$ | $a_{-k}$ |
| Time Scaling | $x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period $mN$) | $\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$ |
| Periodic Convolution | $\sum_{r=\langle N\rangle} x[r]y[n-r]$ | $Na_k b_k$ |
| Multiplication | $x[n]y[n]$ | $\sum_{l=\langle N\rangle} a_l b_{k-l}$ |
| First Difference | $x[n] - x[n-1]$ | $(1 - e^{-jk(2\pi/N)})a_k$ |
| Running Sum | $\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$ | $\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$ |
| Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | $x[n]$ real and even | $a_k$ real and even |
| Real and Odd Signals | $x[n]$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | $\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [x[n] \text{ real}] \end{cases}$ | $\mathcal{Re}\{a_k\}$<br>$j\mathcal{Im}\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

## 4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

**TABLE 4.1** PROPERTIES OF THE FOURIER TRANSFORM

| Section | Property | Aperiodic signal | Fourier transform |
|---|---|---|---|
| | | $x(t)$<br>$y(t)$ | $X(j\omega)$<br>$Y(j\omega)$ |
| 4.3.1 | Linearity | $ax(t) + by(t)$ | $aX(j\omega) + bY(j\omega)$ |
| 4.3.2 | Time Shifting | $x(t - t_0)$ | $e^{-j\omega t_0}X(j\omega)$ |
| 4.3.6 | Frequency Shifting | $e^{j\omega_0 t}x(t)$ | $X(j(\omega - \omega_0))$ |
| 4.3.3 | Conjugation | $x^*(t)$ | $X^*(-j\omega)$ |
| 4.3.5 | Time Reversal | $x(-t)$ | $X(-j\omega)$ |
| 4.3.5 | Time and Frequency Scaling | $x(at)$ | $\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$ |
| 4.4 | Convolution | $x(t) * y(t)$ | $X(j\omega)Y(j\omega)$ |
| 4.5 | Multiplication | $x(t)y(t)$ | $\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$ |
| 4.3.4 | Differentiation in Time | $\frac{d}{dt}x(t)$ | $j\omega X(j\omega)$ |
| 4.3.4 | Integration | $\int_{-\infty}^{t} x(t)dt$ | $\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$ |
| 4.3.6 | Differentiation in Frequency | $tx(t)$ | $j\frac{d}{d\omega}X(j\omega)$ |
| 4.3.3 | Conjugate Symmetry for Real Signals | $x(t)$ real | $\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$ |
| 4.3.3 | Symmetry for Real and Even Signals | $x(t)$ real and even | $X(j\omega)$ real and even |
| 4.3.3 | Symmetry for Real and Odd Signals | $x(t)$ real and odd | $X(j\omega)$ purely imaginary and odd |
| 4.3.3 | Even-Odd Decomposition for Real Signals | $x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]<br>$x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real] | $\mathcal{R}e\{X(j\omega)\}$<br>$j\mathcal{I}m\{X(j\omega)\}$ |
| 4.3.7 | Parseval's Relation for Aperiodic Signals<br>$\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega$ | | |

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**TABLE 4.2** BASIC FOURIER TRANSFORM PAIRS

| Signal | Fourier transform | Fourier series coefficients (if periodic) |
|---|---|---|
| $\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$ | $a_k$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ | $a_1 = 1$<br>$a_k = 0$, otherwise |
| $\cos \omega_0 t$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | $a_1 = a_{-1} = \frac{1}{2}$<br>$a_k = 0$, otherwise |
| $\sin \omega_0 t$ | $\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | $a_1 = -a_{-1} = \frac{1}{2j}$<br>$a_k = 0$, otherwise |
| $x(t) = 1$ | $2\pi\delta(\omega)$ | $a_0 = 1,\ a_k = 0,\ k \neq 0$<br>(this is the Fourier series representation for any choice of $T > 0$) |
| Periodic square wave<br>$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$<br>and<br>$x(t + T) = x(t)$ | $\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$ | $\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$ |
| $\sum_{n=-\infty}^{+\infty} \delta(t - nT)$ | $\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ | $a_k = \frac{1}{T}$ for all $k$ |
| $x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$ | $\frac{2 \sin \omega T_1}{\omega}$ | — |
| $\frac{\sin Wt}{\pi t}$ | $X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$ | — |
| $\delta(t)$ | $1$ | — |
| $u(t)$ | $\frac{1}{j\omega} + \pi\delta(\omega)$ | — |
| $\delta(t - t_0)$ | $e^{-j\omega t_0}$ | — |
| $e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$ | $\frac{1}{a + j\omega}$ | — |
| $te^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$ | $\frac{1}{(a + j\omega)^2}$ | — |
| $\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$,<br>$\mathcal{R}e\{a\} > 0$ | $\frac{1}{(a + j\omega)^n}$ | — |

**TABLE 5.1** PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

| Section | Property | Aperiodic Signal | Fourier Transform |
|---|---|---|---|
| | | $x[n]$<br>$y[n]$ | $\left.\begin{array}{l} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{array}\right\}$ periodic with period $2\pi$ |
| 5.3.2 | Linearity | $ax[n] + by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ |
| 5.3.3 | Time Shifting | $x[n - n_0]$ | $e^{-j\omega n_0}X(e^{j\omega})$ |
| 5.3.3 | Frequency Shifting | $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 5.3.4 | Conjugation | $x^*[n]$ | $X^*(e^{-j\omega})$ |
| 5.3.6 | Time Reversal | $x[-n]$ | $X(e^{-j\omega})$ |
| 5.3.7 | Time Expansion | $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$ | $X(e^{jk\omega})$ |
| 5.4 | Convolution | $x[n] * y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 5.5 | Multiplication | $x[n]y[n]$ | $\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |
| 5.3.5 | Differencing in Time | $x[n] - x[n-1]$ | $(1 - e^{-j\omega})X(e^{j\omega})$ |
| 5.3.5 | Accumulation | $\sum_{k=-\infty}^{n} x[k]$ | $\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$<br>$+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$ |
| 5.3.8 | Differentiation in Frequency | $nx[n]$ | $j\frac{dX(e^{j\omega})}{d\omega}$ |
| 5.3.4 | Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{Re}\{X(e^{j\omega})\} = \mathcal{Re}\{X(e^{-j\omega})\} \\ \mathcal{Im}\{X(e^{j\omega})\} = -\mathcal{Im}\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$ |
| 5.3.4 | Symmetry for Real, Even Signals | $x[n]$ real an even | $X(e^{j\omega})$ real and even |
| 5.3.4 | Symmetry for Real, Odd Signals | $x[n]$ real and odd | $X(e^{j\omega})$ purely imaginary and odd |
| 5.3.4 | Even-odd Decomposition of Real Signals | $x_e[n] = \mathcal{Ev}\{x[n]\}$ $[x[n]$ real$]$<br>$x_o[n] = \mathcal{Od}\{x[n]\}$ $[x[n]$ real$]$ | $\mathcal{Re}\{X(e^{j\omega})\}$<br>$j\mathcal{Im}\{X(e^{j\omega})\}$ |
| 5.3.9 | Parseval's Relation for Aperiodic Signals<br>$\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$ | | |

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

### 5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients $a_k$ of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence $a_k$ in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence $a_k$ are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

**TABLE 5.2** BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k$ |
| $e^{j\omega_0 n}$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\cos \omega_0 n$ | $\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\sin \omega_0 n$ | $\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi r}{N}$<br>$a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $x[n] = 1$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$ | $a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ |
| Periodic square wave<br>$x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & N_1 < \|n\| \le N/2 \end{cases}$<br>and<br>$x[n+N] = x[n]$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \quad k \ne 0, \pm N, \pm 2N, \ldots$<br>$a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots$ |
| $\sum_{k=-\infty}^{+\infty} \delta[n - kN]$ | $\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{1}{N}$ for all $k$ |
| $a^n u[n], \quad \|a\| < 1$ | $\frac{1}{1 - ae^{-j\omega}}$ | — |
| $x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & \|n\| > N_1 \end{cases}$ | $\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$ | — |
| $\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$<br>$0 < W < \pi$ | $X(\omega) = \begin{cases} 1, & 0 \le \|\omega\| \le W \\ 0, & W < \|\omega\| \le \pi \end{cases}$<br>$X(\omega)$ periodic with period $2\pi$ | — |
| $\delta[n]$ | $1$ | — |
| $u[n]$ | $\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$ | — |
| $\delta[n - n_0]$ | $e^{-j\omega n_0}$ | — |
| $(n+1)a^n u[n], \quad \|a\| < 1$ | $\frac{1}{(1 - ae^{-j\omega})^2}$ | — |
| $\frac{(n + r - 1)!}{n!(r-1)!} a^n u[n], \quad \|a\| < 1$ | $\frac{1}{(1 - ae^{-j\omega})^r}$ | — |