Midterm #2 of ECE301, Prof. Wang's section
6:30-7:30pm Wednesday, October 13, 2010, EE 170,

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

E-mail:

Signature:

Question 1: [26%, Work-out question]

1. [4%, Outcome 1] What does the acronym "LTI" stand for? What is the definition of an impulse response?

Consider the following discrete-time LTI system: When the input is a unit step function $x[n] = \mathcal{U}[n]$, the output is

$$y[n] = \begin{cases} 0 & \text{if } n < 0 \\ 2 - (0.5)^n & \text{if } n \geq 0 \end{cases}. \tag{1}$$

2. [10%, Outcome 3] What is the impulse response $h[n]$ of this system?

3. [12%, Outcome 3] When the input is

$$x_1[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } 0 \leq n < 1000 \\ 2 & \text{if } 1000 \leq n \end{cases}, \tag{2}$$

find out the corresponding output $y_1[n]$. (If you do not know the answer to the previous sub-question, you can assume that $h[n] = e^{-n}\mathcal{U}[n]$. You will still get full credit if your answer is correct.)

1. Linear Time-Invariant. The output of the system to an impulse.

2. Let $z[n] = x[n] - x[n-1] = \delta[n]$

$$h[n] = y[n] - y[n-1] = (2-(\tfrac{1}{2})^n)\mathcal{U}[n] - (2-(\tfrac{1}{2})^{n-1})\mathcal{U}[n-1]$$
$$= 1\delta[n] + [(2-(\tfrac{1}{2})^n) - (2-(\tfrac{1}{2})^{n-1})]\mathcal{U}[n-1]$$
$$= 1\delta[n] + [-(\tfrac{1}{2})^n + (\tfrac{1}{2})^{n-1}]\mathcal{U}[n-1]$$
$$= 1\delta[n] + (\tfrac{1}{2})^n(2-1)\mathcal{U}[n-1] = 1\delta[n] + (\tfrac{1}{2})^n\mathcal{U}[n-1] = (\tfrac{1}{2})^n\mathcal{U}[n]$$

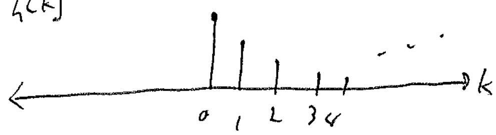
3. $x_1[n] = \mathcal{U}[n] + \mathcal{U}[n-1000] = x[n] - x[n-1000]$

$$y_1[n] = y[n] + y[n-1000]$$
$$= (2-(\tfrac{1}{2})^n)\mathcal{U}[n] + (2-(\tfrac{1}{2})^{n-1000})\mathcal{U}[n-1000]$$

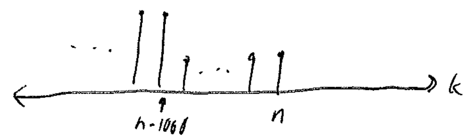
3. (Alt) $h[n] = e^{-n} u[n]$ $\qquad x[n] = u[n] + u[n-1000]$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

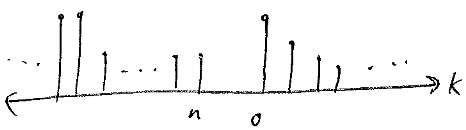
$h[k]$



$x[n-k]$

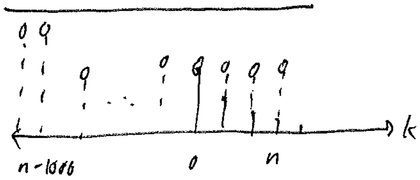


Case 1: $n<0$



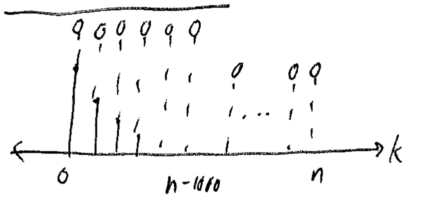
$$y[n] = 0$$

Case 2: $0 \le n < 1000$



$$y[n] = \sum_{k=0}^{n} (1) e^{-k} = \frac{1-e^{-(n+1)}}{1-\frac{1}{e}}$$

Case 3: $n \ge 1000$



$$y[n] = \sum_{k=0}^{n-1000} (2) e^{-k} + \sum_{k=n-999}^{n} (1) e^{-k}$$

$l = k+999$

$k = l-999$

$$= 2\,\frac{1-e^{-(n-999)}}{1-\frac{1}{e}} + \sum_{l=0}^{n+999} e^{-k}$$

$$= 2\,\frac{1-e^{-(n-999)}}{1-\frac{1}{e}} + \frac{1-e^{-(n+1000)}}{1-\frac{1}{e}}$$

$$y[n] = \begin{cases} \dfrac{1-e^{-(n+1)}}{1-\frac{1}{e}} & 0 \le n < 1000 \\[2ex] \dfrac{2\left(1-e^{-(n-999)}\right) + 1-e^{-(n+1000)}}{1-\frac{1}{e}} & n \ge 1000 \\[2ex] 0 & n<0 \end{cases}$$

Question 2: [16%, Work-out question, Outcomes 3 and 5] Consider a continuous-time LTI system with the impulse response being

$$h(t) = \begin{cases} e^{6t} & \text{if } t < 0 \\ 0 & \text{if } t \geq 0 \end{cases}. \tag{3}$$

Consider the following two different inputs: $x_1(t) = e^{j6t}$ and $x_2(t) = e^{j6\sqrt{3}t}$, and denote the corresponding outputs by $y_1(t)$ and $y_2(t)$. Find out the magnitudes $|y_1(1)|$ and $|y_2(1)|$ at time $t = 1$. Find out the phases of $\angle y_1(1)$ and $\angle y_2(1)$ at time $t = 1$.

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \int_{-\infty}^{0} e^{6t} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{t(6-j\omega)} dt = \frac{1}{6-j\omega} e^{t(6-j\omega)} \Big|_{-\infty}^{0}$$

$$= \frac{1}{6-j\omega}$$

$$y_1(t) = H(j6) e^{j6t} = \frac{1}{6-j6} e^{j6t}$$

$$|y_1(1)| = \frac{1}{|6-j6|} = \frac{1}{\sqrt{36+36}} = \frac{1}{\sqrt{72}} = \frac{1}{6\sqrt{2}}$$

$$\angle y_1(1) = 6(1) - \tan^{-1}\left(\frac{-6}{6}\right) = 6 + \frac{\pi}{4}$$

$$y_2(t) = H(j6\sqrt{3}) e^{j6\sqrt{3}t} = \frac{1}{6-j6\sqrt{3}} e^{j6\sqrt{3}t}$$

$$|y_2(1)| = \frac{1}{|6-j6\sqrt{3}|} = \frac{1}{\sqrt{36+3(36)}} = \frac{1}{12}$$

$$\angle y_2(1) = 6\sqrt{3}(1) - \tan^{-1}\left(-\frac{6\sqrt{3}}{6}\right) = 6\sqrt{3} + \frac{\pi}{3}$$

Question 3: [24%, Work-out question] Consider a discrete time LTI system, of which the input/output relationship is

$$y[n] = x[n-1] + y[n-4] \tag{4}$$

and the system is "initial rest," i.e., before we apply any input, the output is zero.

1. [12%, Outcome 2] Find out the impulse response $h[n]$ of this system.

2. [12%, Outcome 4] Find out the Fourier series representation of $h[n]$. (If you do not know the answer to the previous sub-question, you can assume that $h[n] = e^{-n}$ for all $0 \leq n \leq 99$ and $h[n]$ is of period 100. You will still get full credit if your answer is correct.)

1. $h[n] = \delta[n-1] + h[n-4]$

$h[n] = 0$ for $n < 1$

$h[1] = 1$

$h[2] = 0$

$h[3] = 0$

$h[4] = 0$

$h[5] = 1$

$\vdots$

$$h[n] = \begin{cases} 1 & n = 4k+1,\ k \in \mathbb{Z}\ \&\ n \geq 0 \\ 0 & \text{ow} \end{cases}$$

2. $h[n]$ is not periodic, so the Fourier Series does not exist.

(Alt) If it were periodic,

$$a_k = \frac{1}{4} \sum_{n=0}^{3} h[n] e^{-jk\frac{2\pi}{4}n} = \frac{1}{4} e^{-j\frac{\pi}{2}k}$$

(Alt) Using $h[n] = e^{-n}$ for $0 \leq n \leq 99$ & $N = 100$,

$$a_k = \frac{1}{100} \sum_{n=0}^{99} e^{-n} e^{-j\frac{\pi}{50}kn} = \frac{1}{100} \sum_{n=0}^{99} \left(e^{-(1+j\frac{\pi}{50}k)}\right)^n$$

$$= \frac{1}{100} \frac{1 - e^{-(1+j\frac{\pi}{50}k)100}}{1 - e^{-(1+j\frac{\pi}{50}k)}} = \frac{1}{100} \frac{1 - e^{-100}\left(e^{j\overset{2}{\cancel{100}}\pi k}\right)}{1 - e^{-(1+j\frac{\pi}{50}k)}}$$

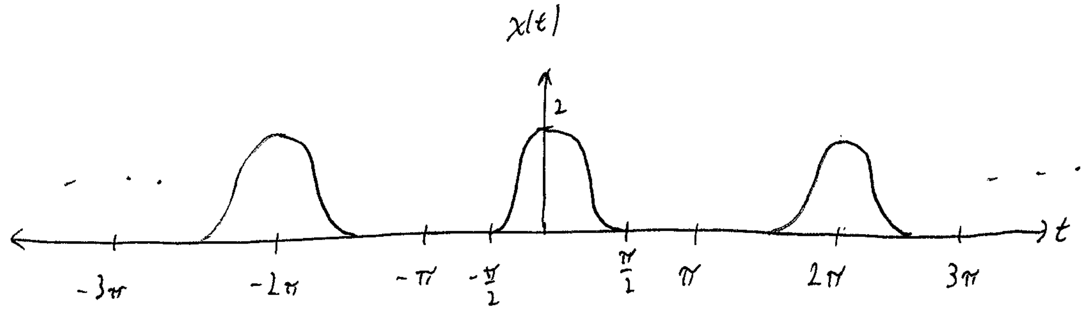
$$= \frac{1}{100} \frac{1 - e^{-100}}{1 - e^{-(1+j\frac{\pi}{50}k)}}$$

Question 4: [26%, Work-out question] Consider the following periodic signal $x(t)$ with period 2π.

$$x(t) = \begin{cases} 1+\cos(2t) & |t| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |t| \le \pi \end{cases}. \tag{5}$$

1. [10%, Outcome 1] Plot $x(t)$ for the range of $-3\pi < t < 3\pi$.
2. [16%, Outcome 4] Find out the Fourier series representation of $x(t)$.

1.



2. $a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-j\frac{2\pi}{2\pi}kt}\,dt = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos(2t))\, e^{-jkt}\,dt$

$= \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-jkt}\,dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(e^{j2t}+e^{-j2t}\right)e^{-jkt}\,dt\right]$

$= \frac{1}{2\pi}\left[\frac{-1}{jk} e^{-jkt}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{jt(2-k)}\,dt + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-jt(2+k)}\,dt\right]$

$= \frac{1}{2\pi}\left[\frac{-1}{jk}\left(e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k}\right) + \frac{1}{2}\frac{1}{j(2-k)} e^{jt(2-k)}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{2}\frac{-1}{j(2+k)} e^{-jt(2+k)}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right]$

$= \frac{1}{2\pi}\left[\frac{2j\sin(\frac{\pi}{2}k)}{jk} + \frac{1}{j2(2-k)}\left(e^{j\pi}e^{-j\frac{\pi}{2}k} - e^{-j\pi}e^{j\frac{\pi}{2}k}\right) - \frac{1}{j2(2+k)}\left(e^{-j\pi}e^{-j\frac{\pi}{2}k} - e^{j\pi}e^{j\frac{\pi}{2}k}\right)\right]$

$= \frac{1}{2\pi}\left[\frac{2\sin(\frac{\pi}{2}k)}{k} + \frac{1}{j2(2-k)}\left(e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right) - \frac{1}{j2(2+k)}\left(e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right)\right]$

$= \frac{\sin(\frac{\pi}{2}k)}{\pi k} + \frac{\sin(\frac{\pi}{2}k)}{2\pi}\left(\frac{1}{2-k} - \frac{1}{2+k}\right)$

$= \frac{\sin(\frac{\pi}{2}k)}{\pi k} + \frac{\sin(\frac{\pi}{2}k)}{2\pi}\left(\frac{2+k-2+k}{4-k^2}\right) = \frac{\sin(\frac{\pi}{2}k)}{\pi k} + \frac{k\sin(\frac{\pi}{2}k)}{\pi(4-k^2)}$

$= \frac{\sin(\frac{\pi}{2}k)}{\pi}\left(\frac{4-k^2+k^2}{k(4-k^2)}\right) = \frac{4\sin(\frac{\pi}{2}k)}{\pi k(4-k^2)}$

Question 5: [10% Outcome 1] The following questions are yes-no questions and there is no need to justify your answers. Consider two LTI systems with impulse responses:

$$\begin{aligned} \text{System 1: } h_1(t) &= 2^{\cos(t)} \\ \text{System 2: } h_2[n] &= \mathcal{U}[n+5] - \mathcal{U}[n] \end{aligned}$$

1. [5%, Outcome 1] Is System 1 stable? Is System 1 causal? Is System 1 memoryless?
2. [5%, Outcome 1] Is System 2 stable? Is System 2 causal? Is System 2 memoryless?

System 1	System 2
NOT stable	Stable
NOT causal	NOT causal
NOT memoryless	NOT memoryless