Question 1: [12.5%, Work-out question] Consider a discrete time signal $x[n]$

$$x[n] = \begin{cases} n & \text{if } 0 \le n \le 2 \\ 3 & \text{if } 3 \le n \le 100 \,. \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Suppose $y[n] = x[-n-1]$.

1. [7.5%] Find the discrete time Fourier transform of $x[n]$.
2. [5%] Find the discrete time Fourier transform of $y[n]$. If you do not know the answer to the first sub-question, you can assume $x[n] = (n+1)2^{-n}\mathcal{U}[n]$ and you will still get full credit if your answer is correct.

1. $X(e^{j\omega}) = 1 \times e^{-j\omega} + 2e^{-j2\omega}$

$+ \sum_{n=3}^{100} 3 \times e^{-j\omega n}$

$= e^{-j\omega} + 2e^{-j2\omega} + \dfrac{3 \times e^{-j3\omega}(1 - e^{-j98\omega})}{1 - e^{-j\omega}}$ #

2. $z[n] = x[n-1]$

$Z(e^{j\omega}) = e^{-j\omega} X(e^{j\omega})$

$y[n] = z[-n] = x[-n-1]$

$= Z(e^{-j\omega}) = e^{j\omega} X(e^{-j\omega})$

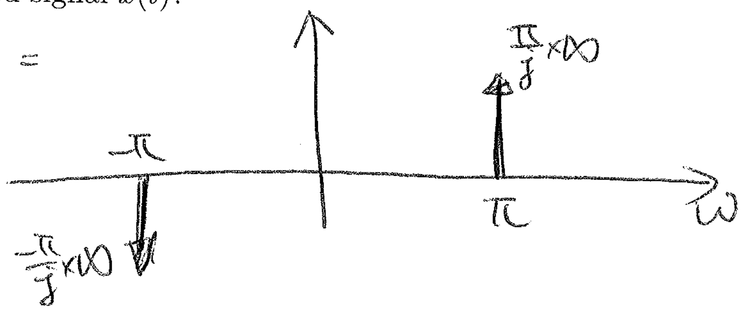
$= e^{j\omega}\left(e^{j\omega} + 2e^{j2\omega} + \dfrac{3 \times e^{j3\omega}(1 - e^{j98\omega})}{1 - e^{j\omega}} \right)$

#

Question 2: [12.5%, Work-out question] Consider a continuous time signal $x(t) = \sin(\pi t)$. We pass it through an impulse train sampler. Let $x_p(t)$ denote the result of impulse train sampling.

1. [2%] Suppose the sampling frequency is 2 Hz. Draw $x_p(t)$ for the range $-1.25 < t < 2.75$.
2. [3%] Continue from the previous question, Draw $X_p(j\omega)$ for the range $-4\pi < \omega < 4\pi$. (The unit of ω is radian/sec, not Hz.)
3. [3%] How do we reconstruct the original signal from $x_p(t)$ using the *optimal reconstruction* (also known as the ideal bandlimited interpolation). You need to specify the cutoff frequencies of your filter(s) and any necessary scaling terms.
4. [2.5%] Suppose the sampling frequency is now reduced to 1 Hz. Draw $X_p(j\omega)$ for the range $-4\pi < \omega < 4\pi$. (The unit of ω is radian/sec, not Hz.)
5. [2%] Continue from the previous question. What is the new $x_p(t)$? And what is the reconstructed signal $\hat{x}(t)$?

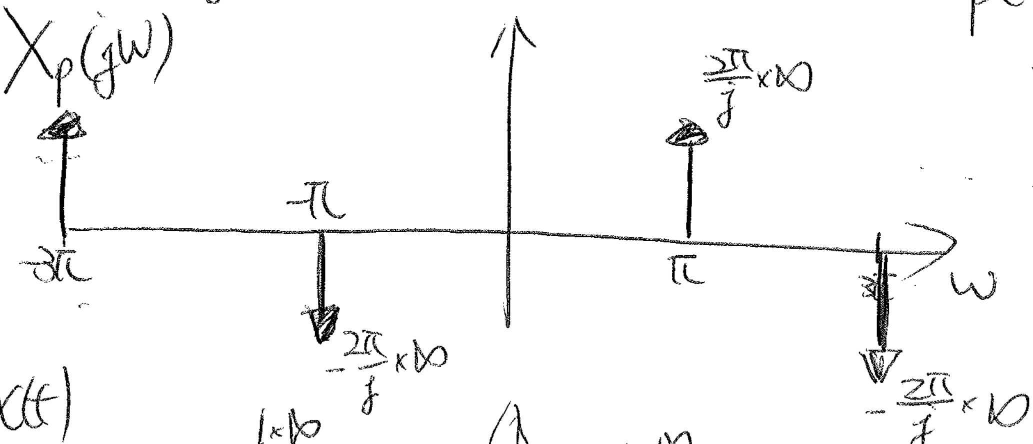
2 $X(j\omega) =$



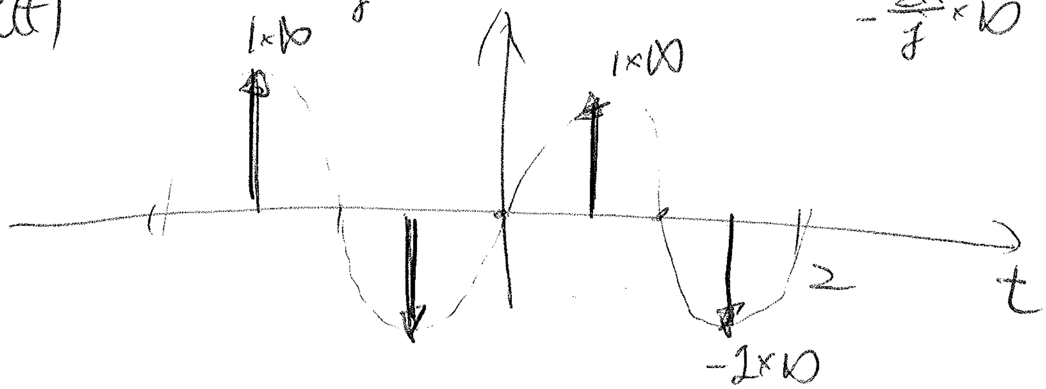
$X_p(jw)$

period $= 4\pi$

$T = 0.5$



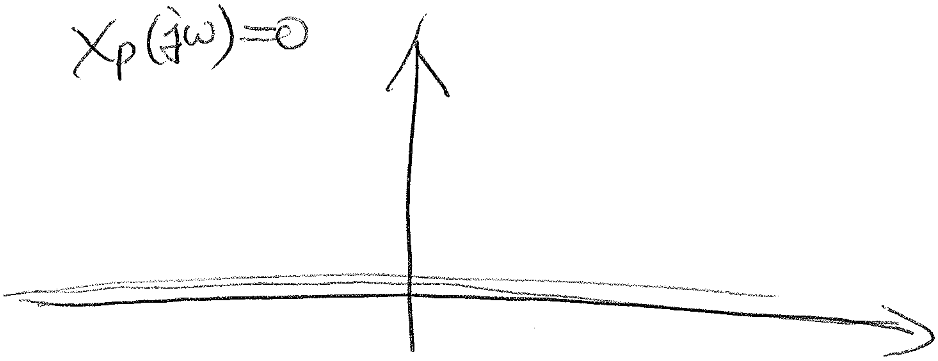
1. $x(t)$



3. A low-pass filter with cut-off freq 2π.

then amplify it by 0.5

4. $X_p(j\omega) = 0$



5. $x_p(t) = 0$ $\hat{x}(t) = 0$.

Question 3: [12.5%, Work-out question] Consider a sampling plus discrete time signal processing system as follows.

For any input signal $x(t)$, it is sampled with sampling frequency 100 Hz, to generate a discrete-time signal $x[n]$. The discrete time signal is then processed through a discrete time linear-time invariant system with impulse response $h_d[n] = 0.5^n \mathcal{U}[n]$. The output is denoted by $y[n]$. Then $y[n]$ is passed through an ideal reconstruction to generate $y(t)$.

Answer the following question:

1. [1.5%] What is the DTFT $H_d(e^{j\omega})$ of $h_d[n]$?
2. [5%] What is the end-to-end frequency response $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$?
3. [3.5%] When the continuous time input is $x(t) = e^{j80\pi t}$, find the overall output $y(t)$.
4. [2.5%] When the continuous time input is $x(t) = e^{j200\pi t}$, find the overall output $y(t)$.

1. $$H_d(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$

2. $$H(j\omega) = \begin{cases} \dfrac{1}{1 - 0.5e^{-j\frac{\omega}{100}}} & \text{if } |\omega| < 100\pi \\ 0 & \text{otherwise} \end{cases}$$

3. $$y(t) = e^{j80\pi t} \cdot \frac{1}{1 - 0.5e^{-j \times 0.8\pi}}$$

4. $y(t) = 0.$

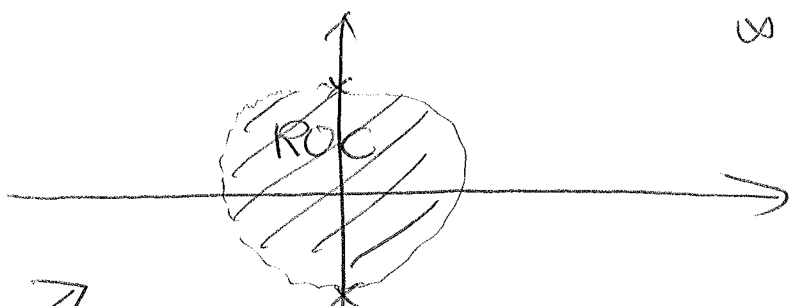
Question 4: [12.5%, Work-out question] Consider a discrete-time signal $x[n]$ with the expression of its Z-transform being $X(z) = \frac{1}{4+z^2} + z$. Suppose we also know that $x[n]$ is left-sided. Answer the following questions.

1. [2.5%] Find out all the poles of $X(z)$ and draw them in a z-plane.
2. [2.5%] What is the ROC of the Z-transform $X(z)$?
3. [2%] Does the discrete-time Fourier transform of $x[n]$ exist?
4. [2%] Is $x[5]$ equal to zero or not?
5. [2%] Find out the value of $x[0]$. You may have to use the time-reversal property before applying the property table.
6. [1.5%] Find out the expression of $x[n]$ by Taylor's expansion.

1. $X(z) = \dfrac{4z + z^3 + 1}{4 + z^2}$

poles: $z = \pm 2j$ & ∞.

∞ is a pole.



2.

3. Yes.

4. ∴ ~~when not considering "z" term~~

~~$X(z)$ does not have~~

∴ zero is not a pole

$\Rightarrow$ $x[5] = 0$

5 $y[n] = x[-n]$

$$Y(z) = \frac{1}{4+z^{-2}} + \frac{1}{z}$$ with poles at $z = \pm \frac{1}{2} j$, $z = 0$

$\therefore \infty$ is not a pole.

$$y[0] = \lim_{z \to \infty} Y(z) = \frac{1}{4}$$

6. ~~X(z)=~~

$$f(x) = \frac{1}{4+x}$$

$$f(0) = \frac{1}{4}$$

$$f'(0) = \frac{1}{(4+x)^2} \times (-1) \Big|_{x=0}$$

$$= \frac{1}{4^2} \times (-1)$$

$$f''(0) = \frac{1}{4^3} \cdots$$

The Taylor's expansion is

$\Rightarrow f(x)$

$$= \sum_{k=0}^{\infty} \frac{1}{4^{k+1}} \times (-1)^k \times \frac{1}{k!} \times x^k$$

$$\Rightarrow x[n] = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n = -1 \\ 0 & \text{if } n < 0 \text{ \& } n \text{ is odd} \\ \frac{1}{4^{|\frac{n}{2}|+1}} \times (-1)^{|\frac{n}{2}|} \times \frac{1}{|\frac{n}{2}|!} & \text{if } n \le 0 \text{ \& even} \end{cases}$$

Question 5: [12.5%, Work-out question] Consider two signals: $x(t) = \mathcal{U}(t+2) - \mathcal{U}(t-2)$ and

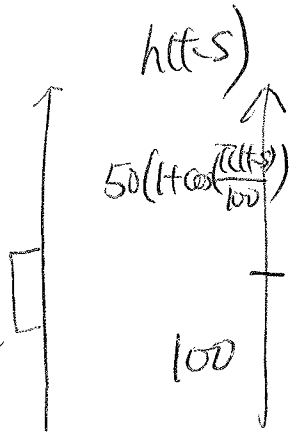
$$h(t) = \begin{cases} 100 & \text{if } 0 < t \\ 50(1+\cos(\frac{\pi t}{100})) & \text{if } t \le 0 \end{cases} \tag{2}$$

Find $y(t) = x(t) * h(t)$.

$$y(t) = \int_{s=-\infty}^{\infty} x(s)\, h(t-s)\, ds.$$

$$h(t-s) = \begin{cases} 100 & \text{if } \cancel{t<s} \quad s<t. \\ 50(1+\cos(\frac{\pi(t-s)}{100})) & \text{if } \cancel{t\ge s} \quad s \ge t \end{cases}$$

$$= \int_{-2}^{2} 1 \cdot h(t-s)\, ds.$$



Case 1. If $t<-2$,

$$y(t) = \int_{-2}^{2} 50\left(1+\cos\left(\frac{\pi(t-s)}{100}\right)\right) ds$$

$$= 50s + \frac{50 \times \sin\left(\frac{\pi(t-s)}{100}\right)}{-\frac{\pi}{100}} \Bigg|_{-2}^{2}$$

$$= 200 + \frac{5000 \times \left[\sin\left(\frac{\pi(t-2)}{100}\right) - \sin\left(\frac{\pi(t+2)}{100}\right)\right]}{-\pi}$$

Case 2: $-2<t<2$

$$= \int_{-2}^{t} 100\, ds + \int_{t}^{2} 50\left(1+\cos\left(\frac{\pi(t-s)}{100}\right)\right) ds$$

$$= 100s\Big|_{-2}^{t} + 50s + \frac{5000 \times \sin\left(\frac{\pi(t-s)}{100}\right)}{-\pi}\Bigg|_{t}^{2}$$

$$= 100t + 200 + \left(100 - 50t + \frac{5000\left(\sin\left(\frac{\pi(t-2)}{100}\right)\right)}{-\pi}\right).$$

$$= 300 + 50t - \frac{5000\left(\sin\left(\frac{\pi(t-2)}{100}\right)\right)}{\pi}$$

Case 3: $2<t$.

$$y(t) = \int_{-2}^{2} \times 100\, ds$$

$$= 100s\Big|_{-2}^{2} = 400 \#$$

Question 6: [20%, Multiple-choice question] Consider two signals $h_1(t) = \cos(\pi t) \cdot \sin(t)$ and $h_2[n] = e^{-n(1+j)}\mathcal{U}[(n-2)^2]$

1. [1%] Is $h_1(t)$ periodic?
2. [1%] Is $h_2[n]$ periodic?
3. [1%] Is $h_1(t)$ even or odd or neither?
4. [1%] Is $h_2[n]$ even or odd or neither?
5. [1%] Is $h_1(t)$ of finite energy?
6. [1%] Is $h_2[n]$ of finite energy?

Suppose the above two signals are also the impulse responses of two LTI systems: System 1 and System 2, respectively.

1. [1%] Is System 1 causal?
2. [1%] Is System 2 causal?
3. [1%] Is System 1 stable?
4. [1%] Is System 2 stable?
5. [1.5%] Is System 1 invertible?
6. [1%] Is System 2 invertible?

1. No.
2. No.
3. odd
4. neither.
5. No.
6. No.
7. No.
8. No.
9. No.
10. No.
11. No
12. Yes (both answers will get credit)

Question 7: [12.5% Work-out question] $\frac{3\pi t}{4}$

1. [6%] $x(t) = \cos(2\pi t) + \sin(3/4\pi t) + e^{j\pi t}$. Find its Fourier series representation.
2. [6.5%] Consider a periodic signal $y[n]$ with period 1000. We also know that

$$y[n] = \begin{cases} 1 & \text{if } 500 \leq n < 1000 \\ 5 & \text{if } 1000 \leq n < 1250 \\ 0 & \text{if } 1250 \leq n < 1500 \end{cases} \tag{3}$$

Find the Fourier series representation of $y[n]$.

1.

$$T_1 = \frac{2\pi}{2\pi} = 1 \qquad T_2 = \frac{2\pi}{3\pi/4} = \frac{8}{3}$$

$$T_3 = \frac{2\pi}{\pi} = 2$$

$$LCM(1, \frac{8}{3}, 2) = 8.$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{+j\frac{2k\pi}{8} \times t}.$$

$$\Rightarrow a_8 = \frac{1}{2} = a_{-8}$$

$$a_3 = \frac{1}{2j} \qquad a_{-3} = \frac{-1}{2j}$$

$$a_4 = 1 \qquad \text{all other } a_k = 0$$

2.

$$a_k = \frac{1}{1000}\left(\sum_{n=500}^{999} \times 1 \times e^{-jk\frac{2\pi}{1000} \times n} + \sum_{n=1000}^{1249} \times 5 \times e^{-jk\frac{2\pi}{1000} \times n}\right)$$

$$= \frac{1}{1000} \times \left[\left(\frac{1 \times e^{-jk\frac{2\pi}{1000}\times 500}\left(1-e^{-jk\frac{2\pi}{1000}\times 500}\right)}{1-e^{-jk\frac{2\pi}{1000}}} \right) + \frac{5\times e^{-jk\times\frac{2\pi}{1000}\times 1000}\left(1-e^{-jk\frac{2\pi}{1000}\times 250}\right)}{1-e^{-jk\frac{2\pi}{1000}}} \right]$$

Question 8: [12.5% Work-out question] Let $z(t) = e^{-2t}\mathcal{U}(t)$. Suppose we know the input/output relationship of a system satisfies $y(t) = \frac{dy(t)}{dt} + z(t) * x(t)$. Find the impulse response of the above system.

$$Y(j\omega) = j\omega \cdot Y(j\omega) + \frac{1}{2+j\omega} \times X(j\omega)$$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{1}{(2+j\omega)(1-j\omega)}$$

$$= \frac{\frac{1}{3}}{2+j\omega} + \frac{\frac{1}{3}}{1-j\omega}$$

$$h(t) = \frac{1}{3}e^{-2t}u(t) + \frac{1}{3}e^{t}u(-t) \quad \#$$