*Question 1:* [25%] Let $x[n] = \cos(\frac{8\pi}{3}n) + \cos(\frac{7\pi}{3}n)$.

1. [4%] Find out the period of $x[n]$.
2. [5%] Find out the Fourier series $a_k$ of $x[n]$.
3. [3%] Plot the Fourier series $a_k$.
4. [6%] Consider an LTI system with $h[n] = 3^{-n}\mathcal{U}[n-1]$. Find out the Fourier transformation $H(e^{j\omega})$ of $h[n]$.
5. [7%] $y[n]$ is the output of the above LTI system with the input being $x[n]$. Find out $y[n]$.

1. $N_1 = \frac{2\pi}{8\pi/3} = \frac{3}{4}$

$N_2 = \frac{2\pi}{7\pi/3} = \frac{6}{7}$

Period $= \text{L.C.M}\left(\frac{3}{4}, \frac{6}{7}, 1\right) = 6$

2. $x[n] = \frac{1}{2}\left(e^{j\frac{8\pi}{3}n} + e^{-j\frac{8\pi}{3}n}\right) + \frac{1}{2}\left(e^{j\frac{7}{3}\pi n} + e^{-j\frac{7}{3}\pi n}\right)$

$= \sum a_k e^{jk\times\frac{2\pi}{6}\times n}$

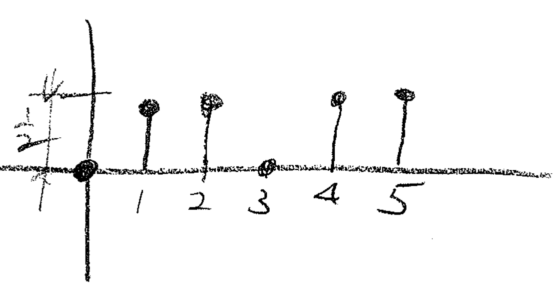
$\Rightarrow a_8 = \frac{1}{2} = a_{-8} \quad a_7 = \frac{1}{2} = a_{-7}$

Or equivalently (move it to $0 \ldots, 5$

$a_2 = \frac{1}{2} = a_4, \quad a_1 = \frac{1}{2} = a_5$

all other $a_k = 0$, $0 \le k \le 5$.

3.



4. $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$

$= \sum_{n=1}^{\infty} 3^{-n} e^{-j\omega n}$

$= \dfrac{\frac{1}{3} e^{-j\omega}}{1 - \frac{1}{3} e^{-j\omega}}$

5. The $b_k$ for $y[n]$ will be.

$b_0 = 0$

$b_1 = a_1 \times H(e^{j \times 1 \times \frac{2\pi}{6}})$

$= \frac{1}{2} \times \dfrac{\frac{1}{3} e^{-j\frac{\pi}{3}}}{1 - \frac{1}{3} e^{-j\frac{\pi}{3}}}$

$b_2 = a_2 \times H(e^{j \times 2 \times \frac{2\pi}{6}})$

$= \frac{1}{2} \dfrac{\frac{1}{3} e^{-j\frac{2\pi}{3}}}{1 - \frac{1}{3} e^{-j\frac{2\pi}{3}}}$

$b_3 = a_3 \times H(\quad) = 0$

$b_4 = \frac{1}{2} \dfrac{\frac{1}{3} e^{-j\frac{4\pi}{3}}}{1 - \frac{1}{3} e^{-j\frac{4}{3}\pi}}$

$b_5 = \frac{1}{2} \dfrac{\frac{1}{3} e^{-j\frac{5}{3}\pi}}{1 - \frac{1}{3} e^{-j\frac{5}{3}\pi}}$

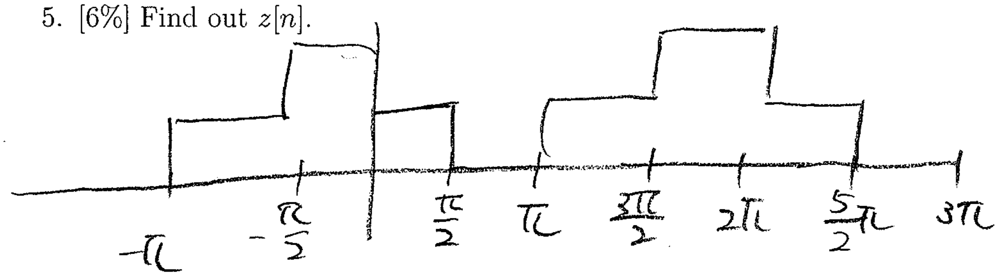
$y[n] = b_1 \times e^{j\frac{\pi}{3}n} + b_2 e^{j\frac{2\pi}{3}n} + b_4 e^{j\frac{4\pi}{3}n} + b_5 \times e^{j\frac{5\pi}{3}n}$ #

*Question 2:* [30%] Consider a discrete time aperiodic signal $x[n]$. Suppose its Fourier transform values within the range $(0, 2\pi)$ are

$$X(e^{j\omega}) = \begin{cases} 1 & \text{if } 0 < \omega < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < \omega < \pi \\ 1 & \text{if } \pi < \omega < \frac{3\pi}{2} \\ 2 & \text{if } \frac{3\pi}{2} < \omega < 2\pi \end{cases} \tag{1}$$

1. [6%] Plot $X(e^{j\omega})$ for $\omega$ ranging from $-\pi$ to $3\pi$.
2. [6%] Suppose $y[n] = x[n] \cdot (-1)^n$. Plot the Fourier transform $Y(e^{j\omega})$ of $y[n]$ for $\omega$ ranging from $-\pi$ to $3\pi$.
3. [6%] What is the value of $y[0]$?
4. [6%] $z[n] = x[n] * y[n]$. Plot $Z(e^{j\omega})$ for $\omega$ ranging from $-\pi$ to $3\pi$.
5. [6%] Find out $z[n]$.

1.

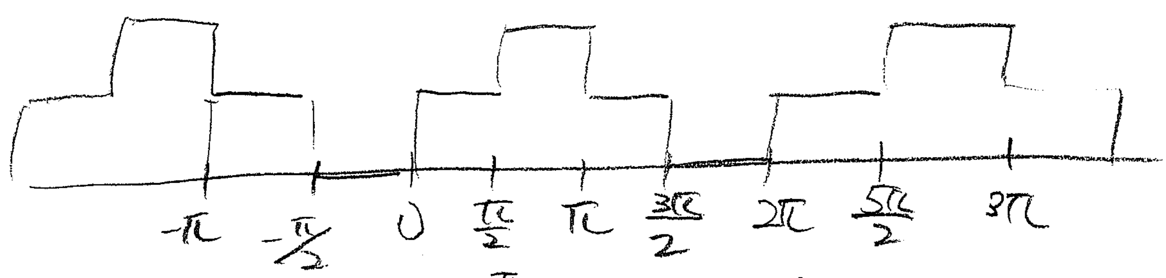



$\because$ periodic with period $2\pi$

2. Since $(-1) = e^{-j\pi}$

$y[n] = x[n] \cdot e^{-j\pi n}$ by freq shift property

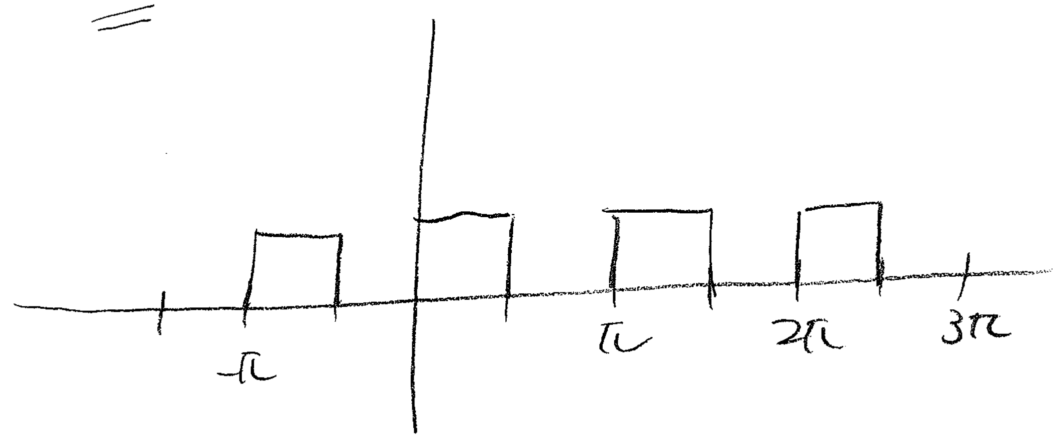
$Y(j\omega) = X(j\omega)$ shift to the left by $\pi$




3. $y[0] = x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega \times 0} \, d\omega = \frac{1}{2\pi} \times (2\pi) = 1$

4. $Z(e^{j\omega}) = X(e^{j\omega}) \times Y(e^{j\omega})$

$=$




5. $z[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(e^{j\omega}) e^{j\omega n} \, d\omega$

$$= \frac{1}{2\pi} \int_{-\pi}^{-\frac{\pi}{2}} e^{j\omega n} \, d\omega$$

$$+ \frac{1}{2\pi} \int_{0}^{\frac{\pi}{2}} e^{j\omega n} \, d\omega$$

$z[0] = \frac{1}{2}$

$\Rightarrow$ $z[n] = \frac{1}{2\pi} \times \frac{e^{-j\frac{\pi}{2}n} - e^{-j\pi n}}{jn}$

$$+ \frac{1}{2\pi} \times \frac{e^{j\frac{\pi}{2}n} - e^{-j \times 0 \times n}}{jn}$$

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*Question 3:* [25%] Consider the following differential equation.

$$12y(t) - 7\frac{d}{dt}y(t) + \frac{d^2}{dt^2}y(t) = 12x(t) - \frac{d}{dt}x(t). \tag{2}$$

1. [8%] Find out the frequency response $H(j\omega)$.
2. [7%] Find out the impulse response $h(t)$.
3. [10%] When the input is $x(t) = 3^{-t}\mathcal{U}(t)$, find out the output $y(t)$. Hint: you need to use Table 4.2 for this problem.

1. $12Y(j\omega) - 7j\omega Y(j\omega) + (j\omega)^2 Y(j\omega)$

$= 12X(j\omega) - (j\omega)X(j\omega)$

$\Rightarrow H(j\omega) = \dfrac{12 - j\omega}{12 - 7j\omega + (j\omega)^2}$

$= \dfrac{12 - j\omega}{(4 - j\omega)(3 - j\omega)}$

$= \dfrac{-8}{4 - j\omega} + \dfrac{9}{3 - j\omega}$

2. $h(t) = -8e^{-}$

This question is designed incorrectly. Full credit for ~~[illegible]~~

$\rightarrow$ (2), (3)

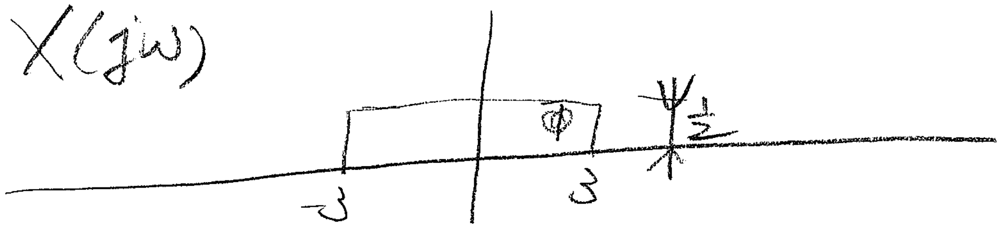
(1) still needs to be graded.

*Question 4:* [20%] Consider a continuous time signal $x(t) = \frac{\sin(3t)}{2\pi t}$.

1. [5%] Plot its Fourier transform $X(j\omega)$ for $\omega$ ranging from $-9$ to 9.
2. [5%] Suppose $y(t) = x(t) j \sin(5t)$. Plot its Fourier transform $Y(j\omega)$ for $\omega$ ranging from $-9$ to 9.
3. [5%] Suppose $z(t) = y(t) j \sin(5t)$. Plot its Fourier transform $Z(j\omega)$ for $\omega$ ranging from $-9$ to 9.
4. [5%] Based on the above answers, describe how to use $\sin(\omega_c t)$ instead of $\cos(\omega_c t)$ to design a very basic AM system. Hint: the output of your base station is $y(t) = x(t) j \sin(5t)$. You are asked to describe how to reconstruct $x(t)$ from $y(t)$. Please do not just divide $y(t)$ by $j\sin(5t)$. In your answer, you should specify the value of the cutoff frequency used in your system.

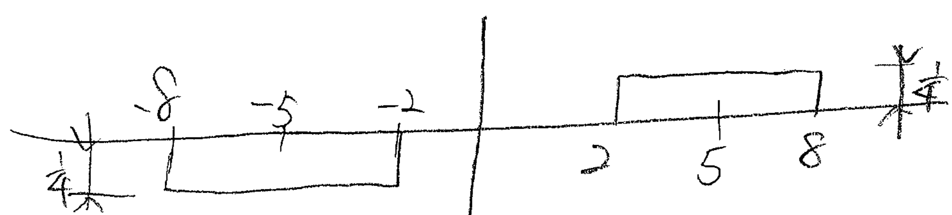
1. By table look-up.

$X(j\omega)$



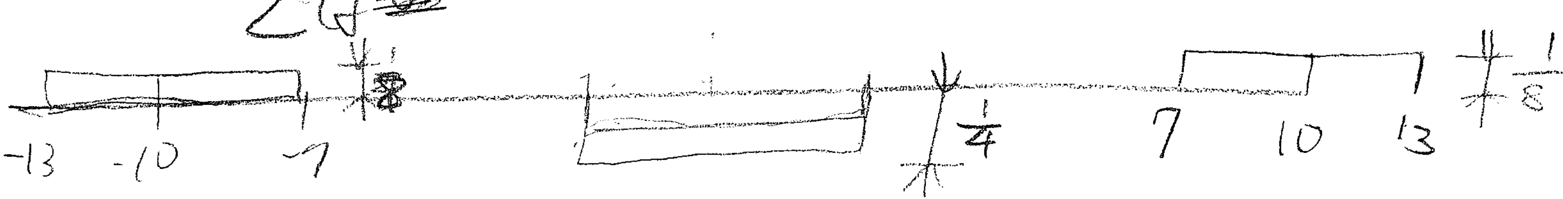

2. $y(t) = x(t) \times j \frac{1}{2j} \left(e^{j5t} - e^{-j5t}\right)$

$= x(t) \times \frac{1}{2} \left(e^{j5t} - e^{-j5t}\right)$




3. $z(t) = y(t) \times j \frac{1}{2j} \left(e^{j5t} - e^{-j5t}\right)$

$Z(j\omega)$



My system will be