Question 1: [22%]

1. [6%] What does the acronym "LTI system" stand for? What is the definition of "impulse response"? What is the definition of "step response"?

2. [4%] Any continuous-time LTI system can be described by its impulse response $h(t)$, where we have $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(s)h(t-s)ds$. What type of "test signals" do we use to derive the above formula for an LTI system?

Consider a continuous-time LTI system as follows.

$$y(t) = x(t-5) + \frac{1}{3}\int_{t-3}^{t} x(s)ds. \tag{1}$$

3. [12%] Find out the impulse response $h(t)$ of this system and plot $h(t)$.

1. Linear time-invariant system
Output when the input is an impulse signal
Output when the input is a step signal.

2. (Shifted versions of) impulse signals

3. $h(t) = \delta(t-5) + \frac{1}{3}\int_{t-3}^{t} \delta(s)ds$

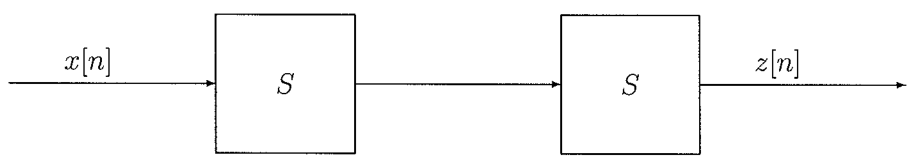
$= \delta(t-5) + \frac{1}{3}\left[u(t) - u(t-3)\right]$

Question 2: [22%] Consider a discrete-time LTI system S described by the following impulse response $h[n]$:

$$h[n] = \delta[n] + \delta[n+1]. \tag{2}$$

1. [10%] Given the input being $x[n] = \frac{1}{(2n-1)^2}$, find out the output $y[n]$.

2. [12%] Two such systems are concatenated serially to form a bigger system with input $x[n]$ and output $z[n]$.



Find out the impulse response $h[n]$ of the new system.

1.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\ &= \sum_{k=-1}^{0} x[n-k] = x[n+1] + x[n] \\ &= \frac{1}{(2(n+1)-1)^2} + \frac{1}{(2n-1)^2} \\ &= \frac{1}{(2n+1)^2} + \frac{1}{(2n-1)^2} \end{aligned}$$

2.

$$\begin{aligned} h[n] &= (\delta[n] + \delta[n+1]) * (\delta[n] + \delta[n+1]) \\ &= \delta[n] + 2\delta[n+1] + \delta[n+2] \end{aligned}$$

Question 3: [36%]

1. [4%] Continuous-time Fourier series representation converts a periodic signal $x(t)$ with period T to a sum of test signals using the following formula:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt.$$

   What are the test signals $x_k(t)$ associated with each coefficient a_k? (Please write-down the mathematic expression of $x_k(t)$.)

2. [6%] Suppose $x(t)$ is periodic with period 6 and

$$x(t) = \begin{cases} \cos(\pi t) & \text{if } -0.5 \le t \le 1.5 \\ 0 & \text{if } 1.5 \le t \le 5.5 \end{cases} \quad (3)$$

   Plot $x(t)$ where t ranges from -4 to 8.

3. [10%] Find out the Fourier series coefficients a_k of $x(t)$. Hint 1: you have to use direct computation. Inspection may not work. Hint 2: rewrite $x(t)$ by its Euler formula, which is easier to integrate.

4. [6%] Suppose $y(t)$ is periodic with period 6 and

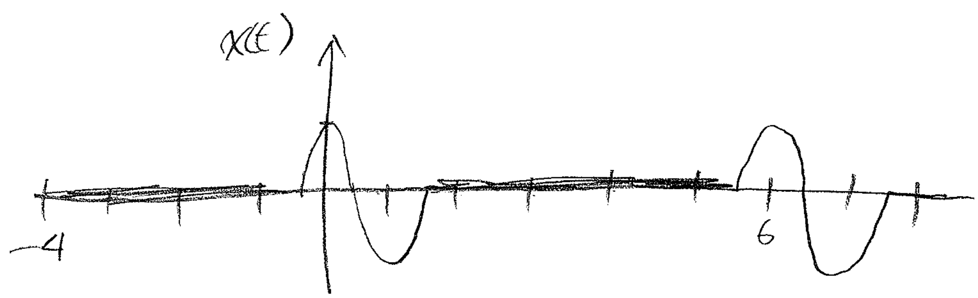
$$y(t) = \begin{cases} \sin(\pi t) & \text{if } 0 \le t \le 4 \\ 0 & \text{if } 4 \le t \le 6 \end{cases} \quad (4)$$

   Plot $y(t)$ where t ranges from -4 to 8.

5. [10%] Find out the Fourier series coefficients b_k of $y(t)$ in terms of a_k. (You do not need to know the answer of a_k. Just write down b_k in terms of a_k.)

1. $x_k(t) = e^{jk\omega_0 t}$ (Harmonically related complex exponentials)

2.



3. $a_0 = \frac{1}{6}\int_{-1}^{5} x(t)\,dt = 0.$

$a_k = \frac{1}{6}\int_{-1}^{5} x(t)e^{-jk\frac{2\pi}{6}t}dt$

$$= \frac{1}{6} \int_{-0,5}^{1,5} \frac{e^{j\pi t} + e^{-j\pi t}}{2} \left(e^{-jk\frac{\pi}{3}t}\right) dt$$

$$= \frac{1}{12} \left[\int_{-0,5}^{1,5} e^{j\pi(1-\frac{k}{3})t} dt + \int_{-0,5}^{1,5} e^{-j\pi(1+\frac{k}{3})t} dt \right.$$

if $k=3$

$$a_3 = \frac{1}{12} \times \left[\int_{-0,5}^{1,5} e^{0} dt + \int_{-0,5}^{1,5} e^{-j \times 2\pi t} dt \right]$$

$$= \frac{1}{12} \times [2] = \frac{1}{6}$$

if $k=-3$

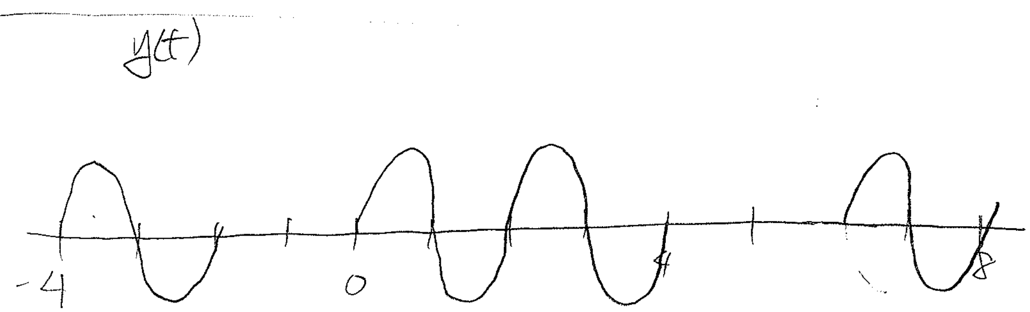
$$a_{-3} = \frac{1}{12} \left[\int_{-0,5}^{1,5} e^{j 2\pi t} dt + \int_{-0,5}^{1,5} e^{j0} dt \right]$$

$$= \frac{1}{6}$$

if $k \neq \pm 3$

$$a_k = \frac{1}{12} \left(\left[\frac{e^{j\pi(1-\frac{k}{3}) \times \frac{3}{2}} - e^{j\pi(1-\frac{k}{3}) \times (-\frac{1}{2})}}{j\pi(1-\frac{k}{3})} \right] + \left[\frac{e^{-j\pi(1+\frac{k}{3})(\frac{3}{2})} - e^{-j\pi(1+\frac{k}{3}) \times (-\frac{1}{2})}}{-j\pi(1+\frac{k}{3})} \right] \right)$$

4



5. $b_k = a_k \left(e^{-jk \times \frac{2\pi}{6} \times (\frac{1}{2})} + e^{-jk \frac{2\pi}{6} \times \frac{3}{2}} \right]$

Question 4: [Multiple Choices, 20%] Consider the following systems:

$$\begin{aligned} \text{System 1: } y(t) &= \int_{s=-\infty}^{\sin(t)} x(s)ds \\ \text{System 2: } y[n] &= \frac{\cos(n+2)+2}{2n-1} x[n] \end{aligned} \quad (5)$$

1. [4%] For Systems 1 and 2, determine whether the systems are memoryless.
2. [4%] For Systems 1 and 2, determine whether the systems are invertible.
3. [4%] For Systems 1 and 2, determine whether the systems are causal.
4. [4%] For Systems 1 and 2, determine whether the systems are linear.
5. [4%] For Systems 1 and 2, determine whether the systems are time-invariant.

Just write down your answers. No need to write down the justification.

S1	S2
W. Memory	Memoryless
Not invertible	Invertible
Non-causal	Causal
Linear	Linear
Time-Varying	Time varying