ECE 301, Final

3:20-5:20pm Monday Dec. 11, PHYS 112,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains 7 questions. For multiple-choice and short-answer questions, there is no need to justify your answers. You have one hour to complete it. I will suggest not spending too much time on a single question, and work on those you know how to solve.
4. The sub-questions of a given question are listed from the easiest to the hardest. The best strategy may be to finish only the sub-questions you know exactly how to solve.
5. There are a total of 17 pages in the exam booklet. Use the back of each page for rough work.
6. **Neither calculator nor crib sheet is allowed.**
7. You are NOT allowed to turn in your exam booklet in the last 5 minutes of the exam.
8. Read through all of the problems first, and consult with the TA during the first 15 minutes. After that, no questions should be asked unless under special circumstances, which is at TA's discretion. You can also get a feel for how long each question might take after browsing through the entire question set. Good luck!

Solution!

Name:

Student ID:

E-mail:

Signature:

Provided $|r| < 1$. The infinite geometric sum formula:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \tag{1}$$

The finite geometric sum formula:

$$\sum_{n=0}^{K} ar^n = \frac{a(1-r^{K+1})}{1-r} \tag{2}$$

The continuous Fourier series pair:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{3}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \tag{4}$$

The continuous Fourier transform pair:

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{5}$$

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{6}$$

The discrete Fourier transform pair:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega \tag{7}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{8}$$

Question 1: [21%]

1. [5%] Consider $x(t)$ as a continuous signal. Is $y(t) = 0.5(x(t) - x(-t))$ is an even or an odd signal? Justify your answer by one sentence. If $x(t) = e^{j3\pi t}$, what is $y(t)$? ($y(t)$ has to be simplified.)

2. [6%] Consider the system taking $x(t)$ as the input and the output $y(t) = 0.5(x(t) - x(-t))$ being the output. Is this system continuous or discrete? Is this system memoryless? Is this system invertible?

3. [6%] Is the above system time-invariant? Is this system linear? What is the corresponding "impulse response" of this system? (For the last sub-question, if you do not know the answer, you may write down the definition of the impulse response instead.)

4. [4%] Consider a discrete signal $x[n]$. We know that it can be decomposed as $x[n] = \sum_{k=-\infty}^{\infty} \alpha_k \delta[n-k]$. What is the expression of α_n in terms of $x[n]$?

1. Odd.

$$y(t) = 0.5(x(t) - x(-t)) = -y(-t)$$
$$= -0.5(x(-t) - x(-(-t)))$$

When $x(t) = e^{j3\pi t}$

$$y(t) = \frac{1}{2}(e^{j3\pi t} - e^{-j3\pi t})$$
$$= j\sin(3\pi t)$$

2. Continuous.

With memory.

Not invertible.

3. Time-varying

Linear

$$h(t) = 0.5(\delta(t) - \delta(-t))$$

4. Since

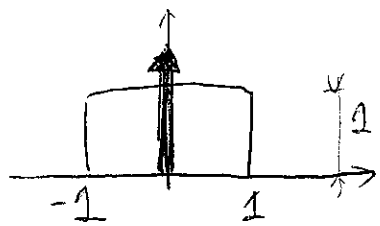
$$x[n] = \alpha_0 \times 0 + \alpha_1 \times 0 + \cdots + \alpha_n \times 1 + \alpha_{n+1} \times 0 + \cdots$$

$$\Rightarrow \alpha_n = x[n] \quad \#$$

Question 2: [15%] Let $x(t) = \delta(t) + \mathcal{U}(t+1) - \mathcal{U}(t-1)$.

1. [3%] Plot $x(t)$.
2. [5%] Consider a continuous moving-average system of window size 2. Namely, $y(t) = \frac{1}{2}\int_{t-2}^{t} x(s)ds$. What is the impulse response $h(t)$?
3. [7%] Find out the expression of the output $y(t)$. (If you do not know how to compute $h(t)$ in the previous sub-question, you can assume $h(t)$ is $\mathcal{U}(t+1) - \mathcal{U}(t-1)$, which is NOT a correct answer. Then proceed by using this wrong $h(t)$.)

1.



2. $h(t) = \frac{1}{2}\int_{t-2}^{t} \delta(s)\,ds$

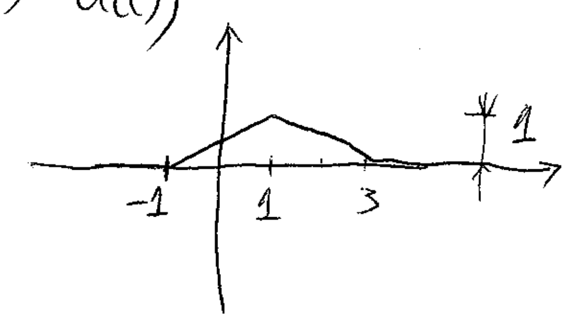
$$= \begin{cases} \frac{1}{2} & \text{if } t-2 < 0 < t \quad \text{(or equivalently } 0 < t < 2\text{)} \\ 0 & \text{otherwise.} \end{cases}$$

$$= \frac{1}{2}(u(t+2) - u(t))$$

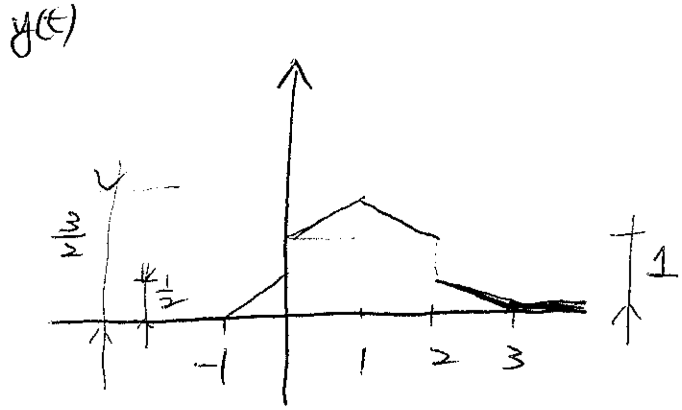
3. We compute $h(t) * \delta(t)$ and $h(t) * (u(t+1) - u(t-1))$ separately.

$h(t) * \delta(t) = h(t) = \frac{1}{2}(u(t+2) - u(t))$

$h(t) * (u(t+1) - u(t-1)) =$



$\Rightarrow h(t) * x(t) =$ (See the next page.


y(t)
3/2
1/2
1
-1
1
2
3

#

Question 3: [20%]

1. [5%] A student A converts a continuous signal $x_A(t)$ into its Fourier series representation: $\alpha_k = 2^{-k}\mathcal{U}[k]$ for all integer k, and $\omega_0 = \frac{2\pi}{3}$ is the fundamental frequency. What is the original signal $x_A(t)$?

2. [5%] Then student A tries to compress the signal by only writing the data of the 0-th to the 9-th coefficients α_k, $0 \le k \le 9$ into a CD-ROM. (Of course, the student also writes the data of ω_0 into the CD-ROM.) Student B tries to use the CD-ROM to reconstruct the original signal. Since there are only 10 coefficients available to student B, he first constructs a series of coefficients β_k by letting

$$\beta_k = \begin{cases} \alpha_k & \text{if } 0 \le k \le 9 \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

   Based on β_k, the student constructs the corresponding signal $x_B(t)$. What is $x_B(t)$?

3. [2%] Professor Wang tries to compare how different are the original signal $x_A(t)$ and the signal $x_B(t)$ constructed from the compressed coefficients β_k. In order to quantify the difference, Professor Want defines the "mean-square distortion" by

$$d \triangleq \int_T |x_A(t) - x_B(t)|^2 dt. \tag{10}$$

   Choose one of the following you think that is true: (a) The larger the d is the more different $x_A(t)$ and $x_B(t)$ are, or (b) the smaller the d is the more different $x_A(t)$ and $x_B(t)$ are.

4. [2%] Use one sentence to describe the Parseval's theorem.

5. [6%] What is the value of d in this example.

1. $T = \frac{2\pi}{\omega_0} = 3$.

$$x_A(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=0}^{\infty} 2^{-k} e^{jk\times\frac{2\pi}{3}t}$$

$$= \frac{1}{1-(2^{-1}\times e^{j\times\frac{2\pi}{3}t})}$$

2. $x_B(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=0}^{9} (2^{-k} e^{jk\times\frac{2\pi}{3}t})$

$$= \frac{1-2^{-10}e^{j\times 10\times\frac{2\pi}{3}\times t}}{1-(2^{-1}e^{j\frac{2\pi}{3}t})}$$

3. (a)

4. Total energy conserves.

5. Let $x_R(t) = x_A(t) - x_B(t)$

We want to find $\int_T |x_R(t)|^2 dt$.

By the linearity, the Fourier series of $x_R(t)$ is

$$c_k = \begin{cases} 2^{-k} & \text{if } k \geq 10 \\ 0 & \text{otherwise.} \end{cases}$$

By the Parseval's theorem

$$\int_T |x_R(t)|^2 = T\times\sum_{k=10}^{\infty}|c_k|^2 = T\times\frac{2^{-20}}{1-2^{-2}}$$

$$\cancel{T\times\frac{2^{-10}}{1-4^{-1}}} = 2^{-18}$$

$$\cancel{= 6\times 2^{-10}}$$

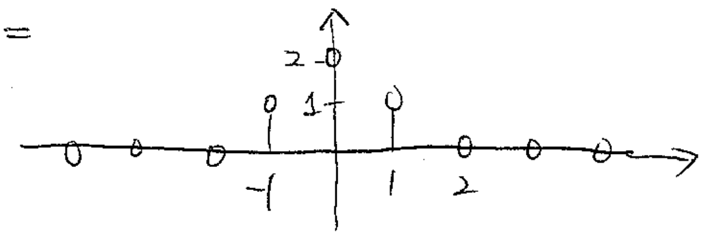
Question 4: [7%]

1. [7%] Consider a discrete signal $x_2[n]$ described as follows.

$$x_2[n] = \begin{cases} n+2 & \text{if } -1 \le n < 0 \\ 2-n & \text{if } 0 \le n < 2 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Find out its Fourier transform $X_2(\omega)$.

Sol^n: $x_2[n] =$



$$X_2(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= e^{-j\omega(-1)} + 2e^{-j\omega\cdot 0} + e^{-j\omega\cdot 1}$$

$$= 2\cos(\omega) + 2$$

Question 5: [8%]

The signal $y(t)$ is generated by convolving a band-limited signal $x_1(t)$ with another band-limited signal $x_2(t)$, that is, $y(t) = x_1(t) * x_2(t)$ where

$$X_1(\omega) = 0 \text{ for } |\omega| > 1500\pi \tag{12}$$

$$X_2(\omega) = 0 \text{ for } |\omega| > 2500\pi. \tag{13}$$

Impulse-train sampling is performed on $y(t)$ to obtain

$$y_p(t) = \sum_{n=-\infty}^{\infty} y(nT)\delta(t - nT). \tag{14}$$

1. [4%] Specify the range of values for the sampling period T, which ensures that $y(t)$ is recoverable from $y_p(t)$.

2. [4%] For perfect reconstruction from the impulse-train sampling, we need to use an ideal low-pass filter. What is the cut-off frequency of the low-pass filter given T being the minimum value such that $y(t)$ is recoverable from $y_p(t)$?

1. Since $Y(\omega) = 0$ for $|\omega| > 1500\pi$

$\Rightarrow \quad W_s > 2W_m = 3000\pi$

$\Rightarrow \quad T = \frac{2\pi}{W_s} < \frac{1}{1500}$

2. The cut-off freq $= \frac{W_s}{2} = 1500\pi$

Question 6: [15%]

1. [2%] $x(t) = \sin(7\pi t)$ is sampled at time instants $t = \frac{n}{6}$ for $n = 0, \pm 1, \pm 2, \ldots$. Let $x_r(t) = \sin(\omega_r t)$ denote the reconstructed signal, for which the cut-off frequency of the low-pass filter is 6π. We notice that $x_r(t) \neq x(t)$. What is the name of this effect?

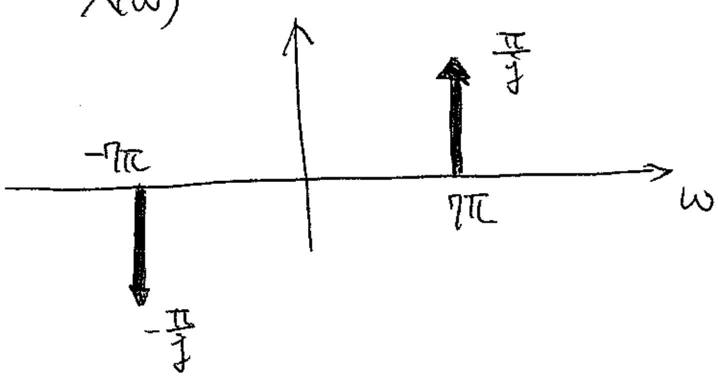
2. [4%] We will solve the value of ω_r step by step. Step 1: Plot $X(\omega)$. Carefully mark the amplitudes / coefficients of the impulses.

3. [4%] Step 2: Plot $X_p(\omega)$, where $X_p(\omega)$ is the Fourier transform of the impulse sampling $x_p(t)$.

4. [5%] [Outcomes 4, 5, and 6: 4%] What is the reconstructed signal $x_r(t)$? What is the value of ω_r?

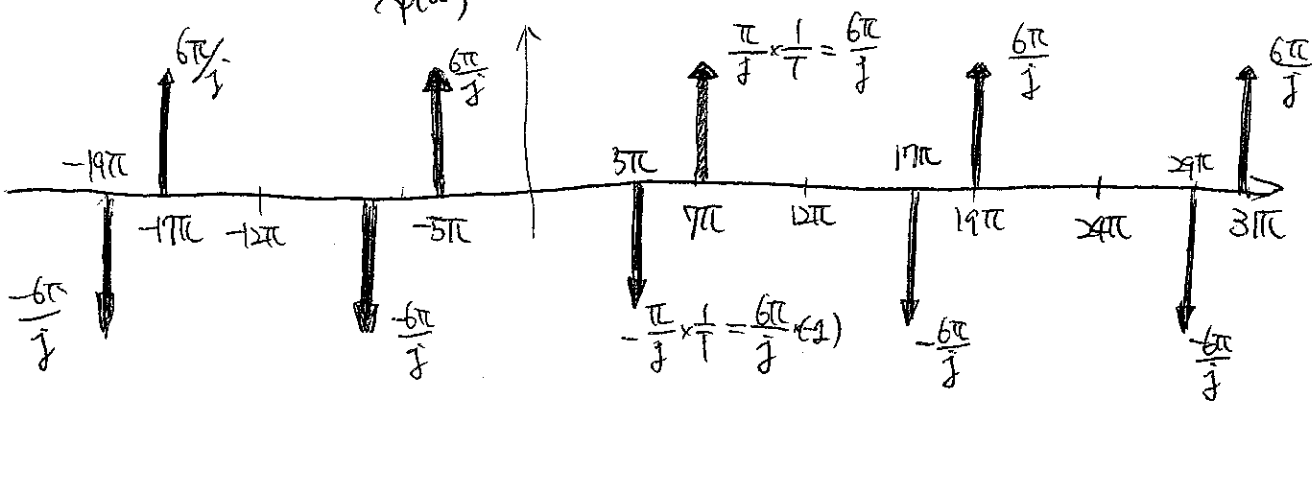
1. Aliasing effect.

2. $X(\omega)$

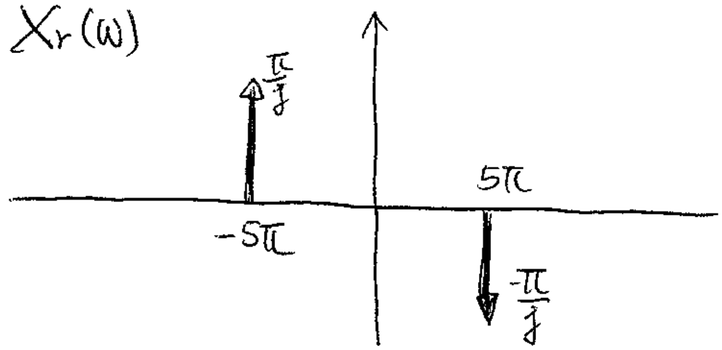


3. $W_s = \frac{2\pi}{T} = \frac{2\pi}{1/6} = 12\pi$

$X_p(\omega)$



For reconstruction, we pass $x_p(t)$ through a LPF with cut-off freq $\frac{\omega_s}{2} = 6\pi$ and gain factor $T = \frac{1}{6}$

$\Rightarrow$ 

$$x_r(t) = \sin(-5\pi t)$$

$\Rightarrow$ $\omega_r = -5\pi$.

Note: This is the reason that on TV you sometimes see a wheel rotate "backward" (negative freq)

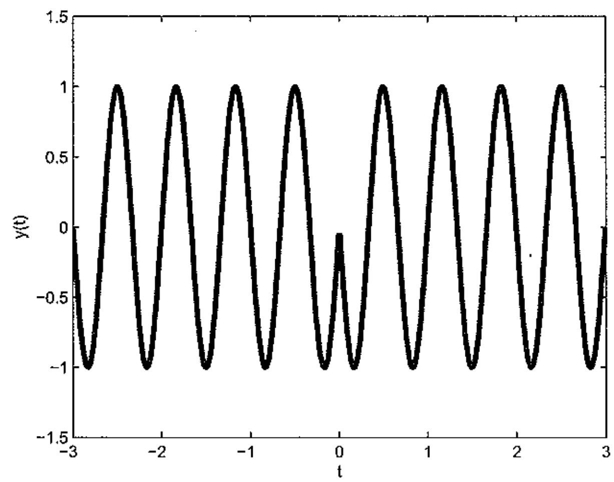
while the car is moving forward.

Question 7: [14%] Consider an AM signal $y(t) = (x(t) + 1.0)\cos(3\pi t)$ is sent by a transmitter.

1. [2%] What type of system is it? Choose among (AM-DSB/WC, AM-DSB/SC, AM-SSB/WC, AM-SSB/SC).

2. [2%] In what condition can we use asynchronous demodulation for the above system so that $x(t)$ can be reconstructed perfectly?

3. [3%] Suppose the receiver receives $y(t)$ as follows. What is the end result if we use asynchronous demodulation to construct the original signal $x(t)$.



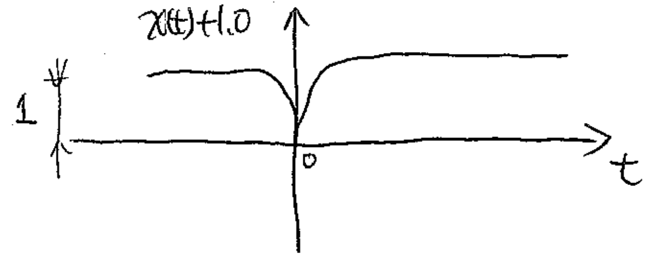
4. [7%] How to construct $x(t)$ from $y(t)$ using synchronous demodulation. Please describe it by a diagram and explicitly specify all the frequencies and gain factors involved.

   Note: There are two different points of demodulating this system when compared to the demodulation of the simple case $x(t)\cos(3\pi t)$. First, we have $y(t) = (x(t) + 1.0)\cos(3\pi t)$ rather than the simple form $x(t)\cos(3\pi t)$. Second, the synchronous carrier at the receiver side is $\sin(3\pi t)$ rather than $\cos(3\pi t)$ due to some time-delay (also known as the phase-shift). You have to do some modifications to the demodulation scheme in the textbook to accommodate these two changes.

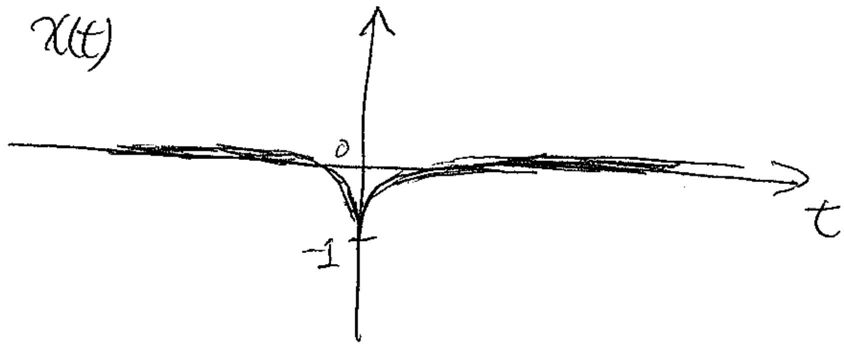
1. AM-DSB/WC

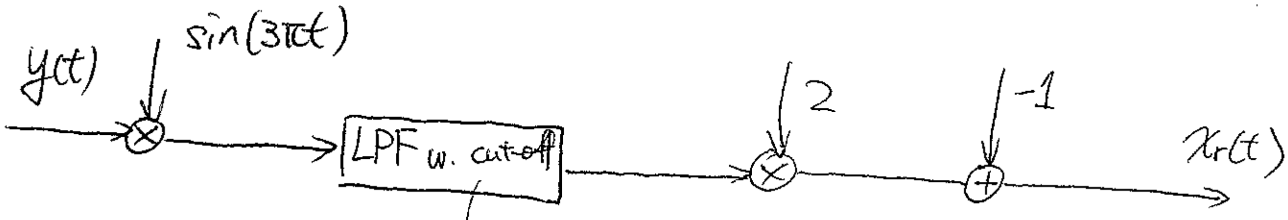
2. $x(t) + 1.0 \geq 0$ for all t

3. $x(t) + 1.0$ will look like



Therefore;





$$\text{cut-off} = \frac{1}{2} \times 6\pi$$

$$= 3\pi$$