ECE 301, Midterm #3

6:30-7:30pm Wednesday, November 12, 2008, PHYS 114,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains only work-out questions. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 14 pages in the exam booklet. Use the back of each page for rough work. The last pages are all the Tables. You may rip the last pages for easier reference. **Do not use your own copy of the Tables. Using your own copy of Tables will be considered as cheating.**
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

E-mail:

Signature:

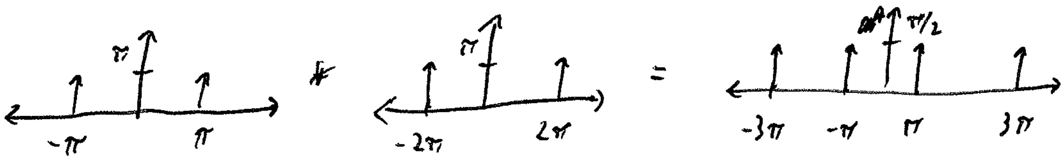
Question 1: [28%] Let $x(t) = \cos(\pi t)$ and $y(t) = \cos(2\pi t)$

1. [10%] Find out the Fourier transforms of $x(t)$ and $y(t)$.
2. [6%] Find out the Fourier transform of $z(t) = x(t) \cdot y(t)$.
3. [6%] Consider a low-pass filter (LPF) with cut-off frequency 2π. Write down the expression of $h(t)$ and $H(j\omega)$.
4. [6%] Let $w(t)$ be the output of the above LPF with the input being $z(t)$. Find out $w(t)$ and $W(j\omega)$. (If you do not know the answer of $z(t)$, you can assume $z(t) = \frac{\sin((3\pi t)/2)}{\pi t} e^{j\frac{3\pi t}{2}}$. You will have full credit if you solve the question correctly using the new $z(t)$.)

1. $x(t) = \frac{1}{2}(e^{j\pi t} + e^{-j\pi t}) \Rightarrow X(j\omega) = \pi\delta(\omega-\pi) + \pi\delta(\omega+\pi)$

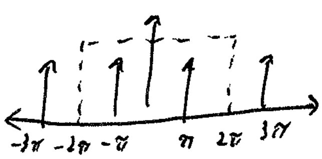
$y(t) = \cos(2\pi t) = \frac{1}{2}(e^{j2\pi t} + e^{-j2\pi t}) \Rightarrow Y(j\omega) = \pi\delta(\omega-2\pi) + \pi\delta(\omega+2\pi)$

2. $z(t) = x(t)y(t) \Rightarrow Z(j\omega) = \frac{X(j\omega) * Y(j\omega)}{2\pi}$



$Z(j\omega) = \frac{\pi}{2}(\delta(\omega+3\pi) + \delta(\omega+\pi) + \delta(\omega-\pi) + \delta(\omega-3\pi))$

3. $H(j\omega) = \begin{cases} 1 & |\omega| \le 2\pi \\ 0 & |\omega| > 2\pi \end{cases}$ $\quad h(t) = \frac{\sin(2\pi t)}{\pi t}$ from table 4.2



4. $W(j\omega) = \frac{\pi}{2}(\delta(\omega+\pi) + \delta(\omega-\pi))$

$w(t) = \frac{1}{2\pi}\frac{\pi}{2}(e^{j\pi t} + e^{-j\pi t}) = \frac{\cos(\pi t)}{2}$

Question 2: [28%] Consider the following discrete time periodic signal $x[n]$ with period 20.

$$x[n] = \begin{cases} e^{j\frac{\pi}{3}n} & \text{if } 0 \le n \le 11 \\ 0 & \text{if } 12 \le n \le 19 \\ x[n-20] & \text{otherwise} \end{cases} \tag{1}$$

1. [10%] Find out the Fourier series coefficients a_k of $x[n]$.
2. [6%] Let $y[n] = e^{-j\frac{\pi}{5}n} + e^{j\frac{\pi}{10}n}$. Find out the Fourier series (period and coefficient b_k) of $y[n]$.
3. [6%] $w[n] = y[n-7]$. Find out the Fourier series (period and coefficient c_k) of $w[n]$.
4. [6%] $z[n] = x[n] \cdot y[n]$. Find out the fifth Fourier series coefficient d_5 of $z[n]$. (If you do not know the Fourier series a_k of $x[n]$, you can express your solution in terms of a_k and you will get 5 points.)

1. $a_k = \frac{1}{20}\sum_{n=0}^{19} x[n] e^{-jk\frac{\pi}{10}n} = \frac{1}{20}\sum_{n=0}^{11} e^{j\frac{\pi}{3}n} e^{-jk\frac{\pi}{10}n} = \frac{1}{20}\sum_{n=0}^{11}\left(e^{j\pi(\frac{1}{3}-\frac{k}{10})}\right)^n$

$= \frac{1}{20}\sum_{n=0}^{11}\left(e^{j\pi(\frac{k}{10}-\frac{1}{3})}\right)^{-n} = \frac{1}{20}\frac{1-\left(e^{-j\pi(\frac{k}{10}-\frac{1}{3})}\right)^{12}}{1-e^{j\pi(\frac{1}{3}-\frac{k}{10})}} = \frac{1}{20}\frac{1-e^{j\pi(\frac{6k}{5}-4)}}{1-e^{j\pi(\frac{1}{3}-\frac{k}{10})}}$

$= \frac{1}{20}\frac{1-e^{j\frac{6\pi}{5}k}}{1-e^{j\frac{\pi}{3}}e^{-j\frac{k\pi}{10}}}$

2. $e^{-j\frac{\pi}{5}(n+N)} = e^{-j\frac{\pi}{5}n} \Rightarrow \frac{\pi}{5}N = k_1 2\pi, \quad N = k_1(10)$

$e^{j\frac{\pi}{10}(n+N)} = e^{j\frac{\pi}{10}n} \Rightarrow \frac{\pi}{10}N = k_2 2\pi, \quad N = k_2(20)$

$\rightarrow N = 20 \qquad \omega_0 = \frac{2\pi}{20} = \frac{\pi}{10}$

$y[n] = e^{-j\frac{2\pi}{10}n} + e^{j\frac{\pi}{10}n}$

$b_k = \begin{cases} 1 & k = 1, -2 \\ 0 & \text{o w} \end{cases}$

3. Table 3.2 : $x[n-n_0] \xleftrightarrow{FS} a_k e^{-jk\frac{2\pi}{N}n_0}$

$N = 20 \qquad c_k = b_k e^{-jk\frac{\pi}{10}7} = \begin{cases} e^{-j\frac{7\pi}{10}} & k = 1 \\ e^{j\frac{14\pi}{10}} & k = -2 \\ 0 & \text{o w} \end{cases}$

4. $z[n] = x[n]\,y[n] \Rightarrow d_k = a_k * b_k$

$$d_k = \sum_{l=0}^{19} a_l\, b_{k-l}$$

$$d_5 = \sum_{l=0}^{19} a_l\, b_{5-l}$$

$5 - l = 1 \Rightarrow l = 4$

$5 - l = -2 \Rightarrow l = 7$

$$= a_4 + a_7$$

$$= \frac{1}{20}\left[\frac{1 - e^{j\frac{6\pi}{5}(4)}}{1 - e^{j\frac{\pi}{3}}e^{-j\frac{\pi}{10}(4)}} + \frac{1 - e^{j\frac{6\pi}{5}(7)}}{1 - e^{j\frac{\pi}{3}}e^{-j\frac{\pi}{10}(7)}}\right]$$

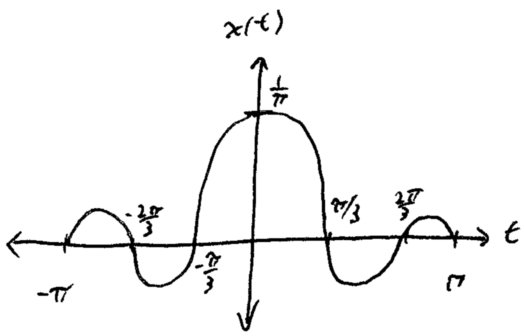
$$= \frac{1}{20}\left[\frac{1 - e^{j\frac{4\pi}{5}}}{1 - e^{j\pi\left(\frac{5-6}{15}\right)}} + \frac{1 - e^{j\frac{2\pi}{5}}}{1 - e^{j\pi\left(\frac{7-3}{30}\right)}}\right]$$

$$\cancel{= \frac{1}{20}\Big[}$$

Question 3: [29%] Consider a continuous time signal $x(t) = \frac{\sin(3t)}{3\pi t}$.

1. [5%] Plot $x(t)$ for $t = -\pi$ to π.
2. [5%] Find out $X(j\omega)$ and plot $X(j\omega)$.
3. [8%] Consider an AM signal $y(t) = x(t)\cos(4t)$. Plot the frequency spectrum $Y(j\omega)$.
4. [6%] Consider a synchronous demodulation scheme. Namely, we let $z(t) = y(t) \cdot 2\cos(\omega_1 t)$ and then pass $z(t)$ through a low-pass filter with cut-off frequency ω_2. Write down a pair of valid choices for the system parameters ω_1 and ω_2.
5. [5%] Suppose we do not have an ideal low-pass filter. Instead, our "practical LPF" has frequency response $H(j\omega) = e^{-|\omega|}$. We use $w(t)$ to denote the final output of passing $z(t)$ through the "practical low-pass filter." Find out $w(t)$. (Hint: Your answer will depend on your previous choices of ω_1 and ω_2.)

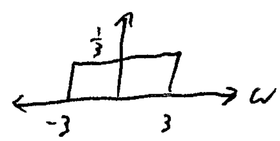
1.



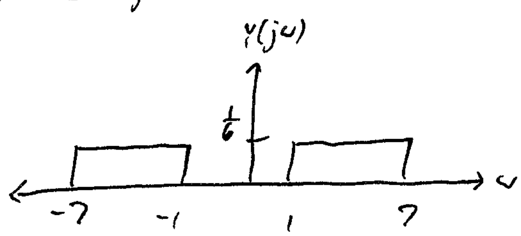
2. Table 4.2:

$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{FT} \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

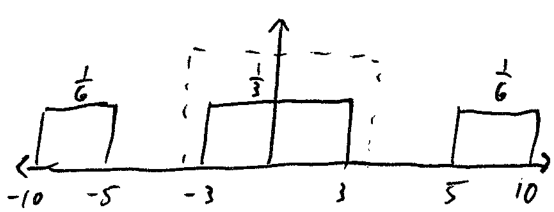
$$H(j\omega) = \begin{cases} \frac{1}{3} & |\omega| < 3 \\ 0 & |\omega| > 3 \end{cases}$$



3. $Y(j\omega) = \frac{1}{2} X(j(\omega - 4)) + \frac{1}{2} X(j(\omega + 4))$



4. $\omega_1 = 4$ $\quad 3 \le \omega_2 \le 5$



5)

$$w(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|} Z(j\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{\omega} Z(j\omega)\, e^{j\omega t}\, d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-\omega} Z(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\Big[\int_{-10}^{-5} \frac{1}{6}\, e^{\omega(1+jt)}\, d\omega + \int_{-3}^{0} \frac{1}{3}\, e^{\omega(1+jt)}$$

$$+ \int_{0}^{3} \frac{1}{3}\, e^{\omega(jt-1)}\, d\omega + \int_{5}^{10} \frac{1}{6}\, e^{\omega(jt-1)}\, d\omega\Big]$$

$$= \frac{1}{2\pi}\Big[\frac{1}{6}\,\frac{1}{1+jt}\, e^{\omega(1+jt)}\Big|_{-10}^{-5} + \frac{1}{3}\,\frac{1}{1+jt}\, e^{\omega(1+jt)}\Big|_{-3}^{0}$$

$$+ \frac{1}{3}\,\frac{1}{jt-1}\, e^{\omega(jt-1)}\Big|_{0}^{3} + \frac{1}{6}\,\frac{1}{jt-1}\, e^{\omega(jt-1)}\Big|_{5}^{10}\Big]$$

$$= \frac{1}{2\pi}\,\frac{1}{3}\Big[\frac{1}{2}\,\frac{1}{1+jt}\left(e^{-5(1+jt)} - e^{-10(1+jt)}\right) + \frac{1}{1+jt}\left(e^{0} - e^{-3(1+jt)}\right)$$

$$+ \frac{1}{jt-1}\left(e^{3(jt-1)} - 1\right) + \frac{1}{2}\,\frac{1}{jt-1}\left(e^{10(jt-1)} - e^{5(jt-1)}\right)\Big]$$

$$= \frac{1}{6\pi}\Big[\frac{1}{jt-1-t^2-jt}\Big]\Big[\frac{1}{2}(jt-1)\left(e^{-5(1+jt)} - e^{-10(1+jt)}\right) + (jt-1)\left(1 - e^{-3(1+jt)}\right)$$

$$+ (jt+1)\left(e^{3(jt-1)} - 1\right) + \frac{1}{2}(jt+1)\left(e^{10(jt-1)} - e^{5(jt-1)}\right)\Big]$$

$$= \frac{-1}{6\pi}\,\frac{1}{t^2+1}\Big[\frac{1}{2}jt\left(e^{-5}\left(e^{-5jt} - e^{j5t}\right) + e^{-10}\left(-e^{-j10t} + e^{j10t}\right)\right)$$

$$+ \frac{1}{2}\left(e^{-5}\left(-e^{-j5t} - e^{j5t}\right) + e^{-10}\left(e^{-j10t} + e^{j10t}\right)\right)$$

$$+ jt\left((1-1) + e^{-3}\left(-e^{-j3t} + e^{j3t}\right)\right)$$

$$+ 1\left((-1-1) + e^{-3}\left(e^{-j3t} + e^{j3t}\right)\right)\Big]$$

$$= \frac{-1}{6\pi}\,\frac{1}{t^2+1}\Big[\frac{1}{2}jt\left(e^{-5}(-2j\sin(5t)) + e^{-10}2j\sin(5t)\right)$$

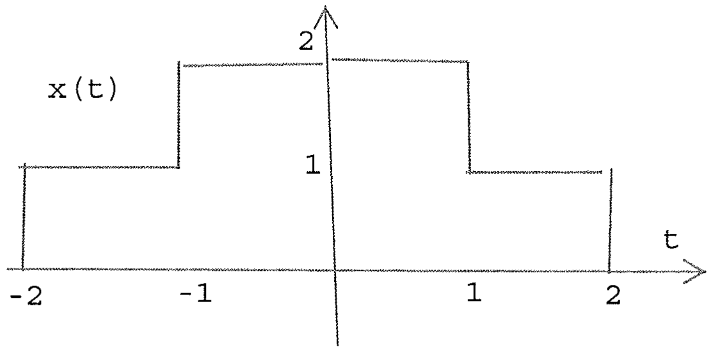
$$+ \frac{1}{2}\left(e^{-5}(-2\cos(5t)) + e^{-10}2\cos(10t)\right)$$

$$+ jt\left(e^{-3}2j\sin(3t)\right) - 2 + 2e^{-3}\cos(3t)\Big]$$

$$= \frac{-1}{6\pi}\,\frac{1}{t^2+1}\Big[te^{-5}\sin(5t) - te^{-10}\sin(10t) - e^{-5}\cos(5t)$$

$$+ e^{-10}\cos(10t) - 2te^{-3}\sin(3t) - 2 + 2e^{-3}\cos(3t)\Big]$$

Question 4: [15%] Consider the following $x(t)$. Answer the following questions. (Hint: Use Tables 4.1 and 4.2.)



1. [3%] $X(j0)$?
2. [3%] $\int_{-\infty}^{\infty} X(j\omega)d\omega$?
3. [3%] $\int_{-\infty}^{\infty} X(j\omega)e^{j\frac{3\omega}{2}}d\omega$?
4. [3%] $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$?
5. [3%] $X(j0.5\pi)$? Hint: Table 4.2.

1. $X(j0) = \int_{-\infty}^{\infty} x(t)\, e^{-j(0)t} dt = \int_{-\infty}^{\infty} x(t)\, dt = 1(1+1) + 2(2) = 2+4 = 6$

2. $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega$

$\Rightarrow \int_{-\infty}^{\infty} X(j\omega)\, d\omega = 2\pi x(0) = 2\pi\,(2) = 4\pi$

3. $\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega\frac{3}{2}} d\omega = 2\pi x(\tfrac{3}{2}) = 2\pi(1) = 2\pi$

4. $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi\int_{-\infty}^{\infty} |x(t)|^2 dt$ by Parseval's Relation

$= 2\pi\left[(1)(1)^2 + (2)(2)^2 + (1)(1)\right] = 2\pi(2+8) = 20\pi$

5. $X(j\tfrac{\pi}{2}) = \int_{-\infty}^{\infty} x(t)\, e^{-j\frac{\pi}{2}t} dt$

$x(t) = x_1(t) + x_2(t)$

$x_1(t) = \begin{cases} 1 & |t|<2 \\ 0 & |t|>2 \end{cases}$ $\quad x_2(t) = \begin{cases} 1 & |t|<1 \\ 0 & |t|>1 \end{cases}$

$X(j\omega) = X_1(j\omega) + X_2(j\omega) = \frac{2\sin(2\omega)}{\omega} + \frac{2\sin(\omega)}{\omega}$

$X(j\tfrac{\pi}{2}) = \frac{2\sin(\pi)}{\frac{\pi}{2}} + \frac{2\sin(\frac{\pi}{2})}{\frac{\pi}{2}} = 0 + \frac{2(1)}{\frac{\pi}{2}} = \frac{4}{\pi}$