ECE 301, Midterm #2

8-9pm Thursday, October 9, 2008, PHYS 114,

1. Enter your name, student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 10 pages in the exam booklet. Use the back of each page for rough work. The last page contains the formula for Fourier series. You may tear the last page for easier reference. **Do not use your own formula sheet or any kind. Using your own formula sheet will be considered as cheating.**
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

E-mail:

Signature:

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(j\omega) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Continuous-time Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Question 1: [24%]

1. [4%] What does the acronym "LTI system" stand for?

2. [6%] Let $x[n]$ and $y[n]$ denote the input and output of a discrete LTI system that has an impulse response $h[n]$. Consider the following two equations:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{1}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{2}$$

   From the perspective of coefficients, test signals, the output of test signals, the weighted sum of signals, and the LTI system, answer the following questions.

   What is the role of $\delta[n-k]$? What is the role of $h[n-k]$? What is the role of $x[k]$?

Consider a continuous-time LTI system as follows.

$$y[n] = \sum_{k=-\infty}^{n} x[k]2^{k-n}. \tag{3}$$

3. [14%] Find out the impulse response $h[n]$ of this system and plot $h[n]$ for the range $n = -2$ to 2.

1. Linear time-invariant

2. $\delta[n-k]$ are the test signals, the shifted delta.

   $h[n-k]$ are the outputs of test signals, the shifted impulse response

   $x[k]$ are the coefficients of the test signals

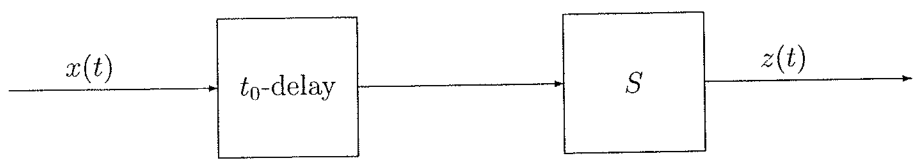
3. $h[n] = \sum_{k=-\infty}^{n} \delta[k]\, 2^{k-n} = \begin{cases} (\frac{1}{2})^n & n \geq 0 \\ 0 & n < 0 \end{cases}$

   $= (\frac{1}{2})^n\, u[n]$

Question 2: [28%] Consider a continuous-time LTI system S described by the following impulse response $h(t)$:

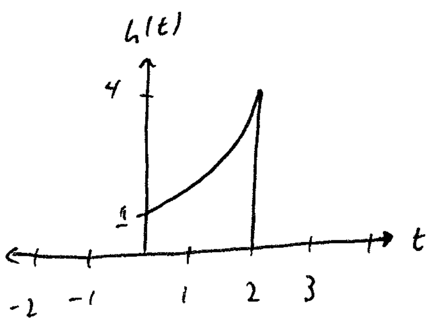
$$h(t) = 2^t(\mathcal{U}(t) - \mathcal{U}(t-2)). \tag{4}$$

1. [4%] Is the system S memoryless? Is the system S causal?
2. [4%] Plot $h(t)$ for the range $t = -2$ to 5.
3. [16%] Given the input being $x(t) = 2^t\mathcal{U}(-t)$, find out the output $y(t)$. Plot $y(t)$.
4. [4%] A system is called "a t_0-delay system" if any input $x(t)$ generates a delayed output $y(t) = x(t - t_0)$. Find out the impulse response of the following serially concatenated system (input being $x(t)$ and output being $z(t)$).



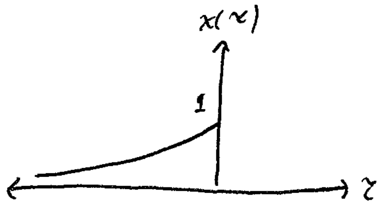
1. S has memory since it needs $x(t)$ for $(t-2, t)$
S is causal since it requires only past & present values of $x(t)$.

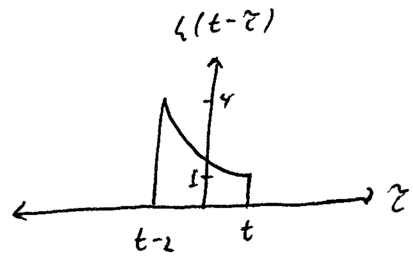
2.



3. $h(t) = 2^t(u(t) - u(t-2))$

$x(t) = 2^t u(-t)$





Region I: $t < 0$

$$y(t) = \int_{t-2}^{t} 2^{\tau} 2^{t-\tau} \, d\tau = 2^t \int_{t-2}^{t} d\tau = 2^{t+1}$$

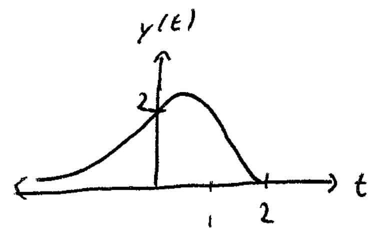
Region II: $0 \le t < 2$

$$y(t) = \int_{t-2}^{0} 2^{\tau} 2^{t-\tau} \, d\tau = 2^t \int_{t-2}^{0} d\tau = 2^t(2-t)$$

Region III: $t \ge 2$

$y(t) = 0$

$$y(t) = \begin{cases} 2^{t+1} & t < 0 \\ 2^t(2-t) & 0 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$



4. $\delta(t) \longrightarrow \boxed{t_0 \text{ delay}} \xrightarrow{y(t)} \boxed{S} \longrightarrow h(t)$

$y(t) = \delta(t - t_0)$

$h(t) = 2^{t-t_0}(u(t-t_0) - u(t-t_0-2))$

Question 3: [28%]

1. [4%] Suppose $y(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$. What is the period of $y(t)$? Plot $y(t)$ where t ranges from -2 to 6.

2. [4%]

   Suppose $x(t)$ is periodic with period 4 and

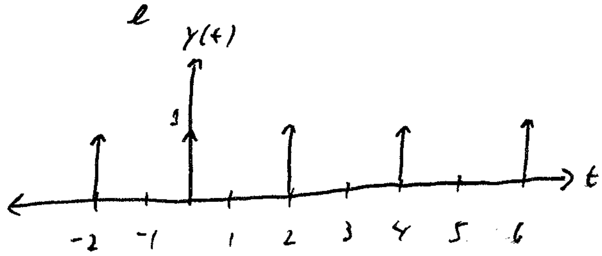
$$x(t) = \begin{cases} 1 - t/3 & \text{if } 0 \le t < 3 \\ t - 3 & \text{if } 3 \le t < 4 \end{cases} \tag{5}$$

   Suppose we also know that $z(t) = x(t) + y(t)$. Plot $z(t)$ where t ranges from -2 to 6.

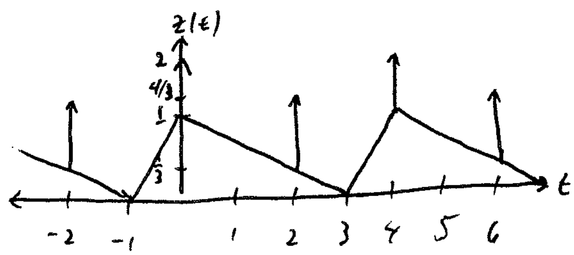
3. [20%] Find out the Fourier series of $z(t)$.

1. $y(t-2) = \sum_k \delta(t-2-2k) = \sum_k \delta(t-(k+1))$ Let $\ell = k+1$

$= \sum_\ell \delta(t-\ell) \Rightarrow T = 2$



2.



3. $\alpha_k = \frac{1}{4}\int_{-1}^{3} z(t) e^{-j\frac{\pi}{2}kt} dt = \frac{1}{4}\int_{-1}^{3} (\delta(t) + \delta(t-2)) e^{-j\frac{\pi}{2}kt} dt + \frac{1}{4}\int_0^3 (1-\frac{t}{3}) e^{-j\frac{\pi}{2}kt} dt$

$+ \frac{1}{4}\int_3^4 (t-3) e^{-j\frac{\pi}{2}kt} dt$

$= \frac{1}{4}(1 + e^{-j\pi k}) + \frac{1}{4}\int_0^3 (1-\frac{t}{3}) e^{-j\frac{\pi}{2}kt} dt + \frac{1}{4}\int_3^4 (t-3) e^{-j\frac{\pi}{2}kt} dt$

$u = 1 - \frac{t}{3}$ $dv = e^{-j\frac{\pi}{2}kt} dt$ $u = t-3$ $dv = \cancel{\frac{1}{j\frac{\pi}{2}k}} e^{-j\frac{\pi}{2}kt} dt$

$du = -\frac{1}{3} dt$ $v = \frac{-1}{j\frac{\pi}{2}k} e^{-j\frac{\pi}{2}kt}$ $du = dt$ $v = \frac{-1}{j\frac{\pi}{2}k} e^{-j\frac{\pi}{2}kt}$

$= \frac{1}{4}(1 + (-1)^k) + \frac{1}{4}\Big[\frac{\frac{t}{3}-1}{j\frac{\pi}{2}k} e^{-j\frac{\pi}{2}kt}\Big|_0^3 + \frac{1}{j\frac{\pi}{2}k}\int_0^3 e^{-j\frac{\pi}{2}kt}(-\frac{1}{3}) dt$

$\frac{3-t}{j\frac{\pi}{2}k} e^{-j\frac{\pi}{2}kt}\Big|_3^4 + \frac{1}{j\frac{\pi}{2}k}\int_3^4 e^{-j\frac{\pi}{2}kt} dt \Big]$

$$\alpha_k = \frac{1}{4}\left(1+(-1)^k\right) + \frac{1}{4}\left[\frac{-1}{j\frac{\pi}{2}k} + \frac{1}{3}\frac{1}{(j\frac{\pi}{2}k)^2}\, e^{-j\frac{\pi}{2}kt}\Big|_0^3\right.$$

$$\left.\frac{-1}{j\frac{\pi}{2}k}\, e^{-j\frac{\pi}{2}(4)} - \frac{1}{(j\frac{\pi}{2}k)^2}\, e^{-j\frac{\pi}{2}kt}\Big|_3^4\right]$$

$$= \frac{1}{4}\left(1+(-1)^k\right) + \frac{1}{4}\left[\frac{-2}{j\frac{\pi}{2}k} - \frac{1}{3}\frac{1}{(\frac{\pi}{2}k)^2}\left(e^{-j\frac{3\pi}{2}k} - 1\right) + \frac{1}{(j\frac{\pi}{2}k)^2}\left(e^{j\frac{\pi}{2}k(4)} - e^{-j\frac{3\pi}{2}k}\right)\right]$$

$$= \frac{1}{4}\left(1+(-1)^k\right) + \frac{1}{4}\left[\frac{-2}{j\frac{\pi}{2}k} - \frac{1}{3}\frac{1}{(\frac{\pi}{2}k)^2}\left(e^{-j\frac{3\pi}{2}k} - 1\right) + \frac{1}{(\frac{\pi}{2}k)^2}\left(1 - e^{-j\frac{3\pi}{2}k}\right)\right]$$

$$= \frac{1}{4}\left(1+(-1)^k\right) + \frac{1}{4}\left(\frac{-4}{j\pi k}\right) + \frac{1}{4(\frac{\pi}{2}k)^2}\left[-\frac{1}{3}e^{-j\frac{3\pi}{2}k} + \frac{1}{3} + 1 - e^{-j\frac{3\pi}{2}k}\right]$$

$$= \frac{1}{4}\left(1+(-1)^k\right) - \frac{1}{j\pi k} + \frac{1}{(\pi k)^2}\left(\frac{4}{3} - \frac{4}{3}e^{-j\frac{3\pi}{2}k}\right)$$

$$= \frac{1}{4}\left(1+(-1)^k\right) - \frac{1}{j\pi k} + \frac{4}{3}\frac{1}{(\pi k)^2}\left(1 - e^{-j\frac{3\pi}{2}k}\right)$$

Question 4: [20%]

1. [8%] Find out the Fourier series of $x(t) = \cos(8t) + \sin(4t)$.

For the remaining questions, just write down your answers. No need to write down the justification.

Consider the following systems:

$$\begin{aligned} \text{System 1: } y(t) &= \begin{cases} x(t)^2 & \text{if } x(t) \geq 0 \\ -x(t) & \text{if } x(t) < 0 \end{cases} \\ \text{System 2: } y[n] &= \sum_{k=-1}^{1} 2^k x[n-k] \end{aligned} \tag{6}$$

1. [4%] For Systems 1 and 2, determine whether the systems are causal.
2. [4%] For Systems 1 and 2, determine whether the systems are linear.
3. [4%] For Systems 1 and 2, determine whether the systems are time-invariant.

1. $x(t) = \cos(8t) + \sin(4t)$ $\qquad T = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\left.\begin{aligned} &= \frac{1}{2}\left(e^{j8t} + e^{-j8t}\right) + \frac{1}{2j}\left(e^{j4t} - e^{-j4t}\right) \\ &= \sum_k a_k e^{j\frac{2\pi}{\pi/2}kt} = \sum_k a_k e^{j4kt} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &a_1 = \frac{1}{2j} \qquad a_{-1} = \frac{-1}{2j} \\ &a_2 = a_{-2} = \frac{1}{2} \\ &a_k = 0 \quad \text{for} \quad k \neq \pm 1, \pm 2 \end{aligned}$$

1. System 1 is causal since it uses only present values of $x(t)$ to compute $y(t)$.
System 2 is non-causal as it requires $x[n+1]$ $(k=-1)$ to compute $y[n]$

2. System 1 is non-linear since $y(t) = (x(t))^2$ for $x(t) \geq 0$.
System 2 is linear.
Let $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$$\begin{aligned} y[n] &= \sum_{k=-1}^{1} 2^k \left(\alpha_1 x_1[n-k] + \alpha_2 x_2[n-k]\right) = \alpha_1 \sum_{k=-1}^{1} 2^k x_1[n-k] + \alpha_2 \sum_{k=-1}^{1} 2^k x_2[n-k] \\ &= \alpha_1 y_1[n] + \alpha_2 y_2[n] \end{aligned}$$

3. System 1 is time-invariant

$$x(t) \longrightarrow y(t) \quad \text{delay} \quad y(t-t_0) = \begin{cases} x^2(t-t_0) & x(t-t_0) \geq 0 \\ -x(t-t_0) & x(t-t_0) < 0 \end{cases}$$

$$x(t-t_0) \longrightarrow \begin{cases} x^2(t-t_0) & x(t-t_0) \geq 0 \\ -x(t-t_0) & x(t-t_0) < 0 \end{cases}$$

equal

System 2 is time-invariant

$$x[n] \rightarrow y[n] \quad \text{delay} \quad y[n-n_0] = \sum_{k=-1}^{1} 2^k x[n-n_0-k]$$

$$x[n-n_0] \longrightarrow \sum_{k=-1}^{1} 2^k x[n-n_0-k]$$

equal