Question 1: [Multiple Choices, 20%] Consider a LTI system with impulse response $h(t) = \mathcal{U}(t) - \mathcal{U}(t-2)$. Consider an input signal $x(t) = \cos(2\pi t) + \sin(\pi t)$ and denote the corresponding output as $y(t)$. For convenience, we let $x_c(t) = \cos(2\pi t)$ and $x_s(t) = \sin(\pi t)$ and $X(\omega)$, $X_c(\omega)$, $X_s(\omega)$, $H(\omega)$, and $Y(\omega)$ are the corresponding the Fourier transforms of $x(t)$, $x_c(t)$, $x_s(t)$, $h(t)$, and $y(t)$ respectively.

1. [Outcome 4, 4%] What is the value of $X_c(\pi)$? (a) 0, (b) 0.5, (c) 0.5∞.
2. [Outcome 4, 4%] What is the value of $X_s(\pi)$? (a) 0, (b) $-0.5j$, (c) $-0.5\infty j$.
3. [Outcomes 2 and 4, 4%] What is the value of $H(0)$? (a) 0, (b) 1, (c) 2.
4. [Outcomes 2, 4, and 5, 4%] What is the value of $Y(1)$? (a) 0, (b) 1, (c) 2.
5. [Outcomes 4, and 5, 4%] How would you name the system? (a) a low-pass filter, (b) a high-pass filter.

1. (a)
2. (c)
3. (c)
4. (a)
5. (a)

Question 2: [Short-Answer, 15%] Please provide a short, one-sentence explanation of the following terms / theorems.

1. [Outcomes 1, 4, and 5, 3%] Fourier series / transformations convert the original signal $x(t)$ (or $x[n]$) to another representation with different "basis signals." What type of signals are the "basis signals" of the Fourier series / transformation representation?

2. [Outcome 4, 3%] Suppose $X(\omega)$ is the Fourier transformation of $x(t)$. What is the physical meaning of $X(0)$? What is the physical meaning of $X(100)$?

3. [Outcome 4, 3%] What is the physical meaning of the Parseval's theorem / relationship?

4. [Outcomes 4, and 5, 3%] An important feature of converting signals to their Fourier representations is that the response $y(t) = h(t) * x(t)$ of a LTI system becomes $Y(\omega) = H(\omega)X(\omega)$. What is the physical meaning of the last equation?

5. [Outcomes 1, 4, and 5, 3%] A LTI system of impulse response $h(t)$ is invertible if and only if its corresponding Fourier transform $H(\omega) \neq 0$ for all ω. Why is it so? Hint: $Y(\omega) = H(\omega)X(\omega)$.

1. sinusoidal waves (complex exponential signals)

2. $X(0)$ = DC component.

   $X(100)$ = The frequency 100 component.

3. The total energy conserves.

4. The output of a sinusoidal wave input is still sinusoidal with amplitude change $|H(\omega)|$ & phase change $\angle H(\omega)$

5. Because we can then construct the inverse freq response as $H_I(\omega) = \frac{1}{H(\omega)}$

   $\Rightarrow H_I(\omega) \cdot Y(\omega) = X(\omega)$

Question 3: [Work-out Question 25%] Consider a discrete signal $x[n]$ of period 5 such that within one period, the values of the signal is

$$x[n] = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 0 & \text{if } n = 3 \\ 0 & \text{if } n = 4 \end{cases}.$$

1. [Outcome 4, 7%] Find out the Fourier series representation of $x[n]$. Note that this question is asking a complete representation so you not only have to specify the coefficient α_k but also have to specify the corresponding frequency.
2. [Outcome 4, 5%] Find out the value of $\sum_{k=0}^{4} \alpha_k$.
3. [Outcome 4, 6%] Find out the value of $\sum_{k=0}^{4} |\alpha_k|^2$. Note: α_k may be a complex number and $|\cdot|$ is the corresponding absolute value of a complex number.
4. [Outcomes 4, and 5, 7%] Let $y[n] = x[n] - x[n-1]$. Find out the Fourier series representation of $y[n]$. Again, one needs to specify both the coefficients β_k and the corresponding frequency.

1. $W_0 = \frac{2\pi}{5}$ $\Rightarrow$ the corresponding frequencies are $k\frac{2\pi}{5}$ for $k = 0, \pm 1, \pm 2,$ ~~...~~

$$\alpha_k = \frac{1}{5} \sum_{n=0}^{4} x[n]\, e^{-jk\frac{2\pi}{5}n}$$

$$= \frac{1}{5} e^{-jk\frac{2\pi}{5}} + \frac{2}{5} e^{-jk\frac{4\pi}{5}}$$

2. $$\sum_{k=0}^{4} \alpha_k = x[0] = 0$$

3. $$\sum_{k=0}^{4} |\alpha_k|^2 = \frac{1}{N} \sum_{n=0}^{4} |x[n]|^2$$

$$= \frac{1}{5}(1^2 + 2^2) = 1$$

4. The corresponding frequencies are $k = 0, \pm 1, \pm 2$

$$\beta_k = \left(1 - e^{-jk\frac{2\pi}{5}\times 1}\right)\alpha_k$$

$$= \left(1 - e^{-jk\frac{2\pi}{5}}\right)\left(\frac{1}{5}e^{-jk\times\frac{2\pi}{5}} + \frac{2}{5}e^{-jk\times\frac{4\pi}{5}}\right)$$

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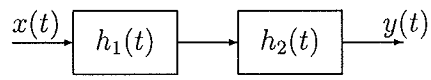
Question 4: [Work-out Question 20+5%] Consider a serial concatenation of two LTI systems as follows.



where $h_1(t) = e^{-3t}\mathcal{U}(t)$ and $h_2(t) = e^{-5t}\mathcal{U}(t)$.

1. [Outcomes 2, 3, 4, and 5, 10%] Let $h(t) = h_1(t) * h_2(t)$ denote the impulse response of this serially concatenated system. Find $H(\omega)$.

2. [Outcomes 4, and 5, 10%] Suppose $x(t) = e^{j2t}$, and we know that the output $y(t) = Ae^{j(2t+\theta)}$. Find out the gain factor A and the phase shift θ. Hint: Let $C = Ae^{j\theta}$ and rewrite $y(t) = Ce^{j2t}$. Now consider input-output relationship in the Fourier domain and find the value of C.

3. [Bonus: Outcomes 4, and 5, 5%] Find $h(t)$. Hint: One can obtain the result from $H(\omega)$ or one can directly compute the convolution.

1. $$H_1(w) = \int_{-\infty}^{\infty} h_1(t)\, e^{-jwt}\, dt$$

$$= \int_{0}^{\infty} e^{-3t} e^{-jwt}\, dt$$

$$= \frac{1}{3+jw}$$

Similarly

$$H_2(w) = \frac{1}{5+jw}$$

$$\Rightarrow H(w) = H_1(w) \cdot H_2(w) = \frac{1}{3+jw} \times \frac{1}{5+jw}$$

2. $$y(t) = H(w)\, e^{j2t} \quad \text{where } w=2$$

$$\Rightarrow A = |H(2)| = \frac{1}{\sqrt{9+4}} \times \frac{1}{\sqrt{25+4}} = \frac{1}{\sqrt{13}} \times \frac{1}{\sqrt{29}} = \frac{1}{\sqrt{377}}$$

$$\theta = -\tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{2}{5}\right) = \angle H(w)$$

3. We notice that

$$H(w) = \frac{1}{2}\left(\frac{1}{3+jw} - \frac{1}{5+jw}\right)$$

$$\Rightarrow h(t) = \frac{1}{2}e^{-3t}u(t) - \frac{1}{2}e^{-5t}u(t)$$

Question 5: [Work-out Question 25%] Consider a discrete moving average system of window size 3: $h[n] = 1/3(\delta[n] + \delta[n-1] + \delta[n-2])$.

1. [Outcomes 4, and 5, 10%] Find out the Fourier transform $H(\omega)$ of $h[n]$. Note: we are considering the discrete Fourier transform.
2. [Outcome 4, 10%] $x[n] = \cos(3n + \pi/2)$. Find out the Fourier transform $X(\omega)$ of $x[n]$.
3. [Outcomes 2, 3, 4, and 5, 5%] What is the output $y[n]$ and its Fourier transform $Y(\omega)$?

1. $$H(w) = \sum_{n=-\infty}^{\infty} h[n] e^{-jwn}$$

$$= \sum_{n=0}^{2} \frac{1}{3} e^{-jwn}$$

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$$= e^{-jw}\left(\frac{1}{3} + \frac{2}{3}\cos(w)\right)$$

2. $$x[n] = \cos(3n + \pi/2)$$

$$= \sin(3n) \times (-1)$$

$$= -\frac{1}{2j}\left(e^{j3n} - e^{-j3n}\right)$$

$$= \frac{1}{2\pi}\int_{2\pi} X(w) e^{jwn}\, dw$$

$$\Rightarrow X(w) = \sum_{m=-\infty}^{\infty} \frac{-\pi}{j}\delta(w-3+2\pi m) + \frac{\pi}{j}\delta(w+3+2\pi m)$$

$$Y(\omega) = H(\omega)X(\omega)$$

$$= \sum_{m=-\infty}^{\infty} H(3)\frac{-\pi}{j}\delta(\omega-3+2\pi m) + H(-3)\frac{\pi}{j}\delta(\omega+3+2\pi m)$$

$$= \sum_{m=-\infty}^{\infty} e^{-3j}\left(\frac{1}{3}+\frac{2}{3}\cos(3)\right)\frac{-\pi}{j}\delta(\omega-3+2\pi m)$$

$$+ e^{3j}\left(\frac{1}{3}+\frac{2}{3}\cos(-3)\right)\frac{\pi}{j}\delta(\omega+3+2\pi m) \quad \#$$

$$y[n] = e^{-3j}\left(\frac{1}{3}+\frac{2}{3}\cos(3)\right)\times\frac{-\pi}{j}\left(e^{j3n}\times\frac{1}{2\pi}\right)$$

$$+ e^{3j}\left(\frac{1}{3}+\frac{2}{3}\cos(-3)\right)\times\frac{\pi}{j}\left(e^{-j3n}\times\frac{1}{2\pi}\right)$$

$$= \left(\frac{1}{3}+\frac{2}{3}\cos(3)\right)\sin(3n-3)\times(-1) \quad \#$$

Or by directly computing the convolution

$$y[n] = \frac{1}{3}\cos\left(3n+\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(3(n-1)+\frac{\pi}{2}\right)$$

$$+ \frac{1}{3}\cos\left(3(n-2)+\frac{\pi}{2}\right) \quad \#$$

$$\cancel{= \frac{1}{3}\cos\left(3n+\frac{\pi}{2}\right) + \frac{1}{3}\cos(3n \ldots)}$$