Question 1: [Multiple Choices, 10%] Consider the following continuous signals

$$
\begin{aligned}
x_1(t) &= \mathcal{E}v\{\cos(2\pi t)\mathcal{U}(t-1)\} \\
x_2(t) &= [\sin(t-\pi/3)]^2 \\
x_3(t) &= \sin(\frac{\pi}{8}t^2) \\
x_4(t) &= e^{j(t-1)} + e^{j(2t-1)}
\end{aligned}
$$

and discrete signals

$$
\begin{aligned}
x_5[n] &= \mathcal{E}v\{\cos(2\pi n)\mathcal{U}[n]\} \\
x_6[n] &= \sin(7n/3 - \pi/3) \\
x_7[n] &= \sin(\frac{\pi}{8}n^2) \\
x_8[n] &= e^{j(n-1)} + e^{j(2n-1)},
\end{aligned}
$$

where $\mathcal{E}v\{x(t)\}$ means the even part of the signal $x(t)$.

1. [Outcome 1, 5%] For $x_1(t)$ to $x_4(t)$, determine whether it is periodic or not. If it is periodic, write down the fundamental period.
2. [Outcomes 1 and 6, 5%] For $x_5[n]$ to $x_8[n]$, determine whether it is periodic or not. If it is periodic, write down the fundamental period.

	periodic ,	T or N	aperiodic
$x_1(t)$			✓
$x_2(t)$	✓	π	
$x_3(t)$			✓
$x_4(t)$	✓	π	
$x_5[n]$			✓
$x_6[n]$			✓
$x_7[n]$	✓	8	
$x_8[n]$			✓

Question 2: [Multiple Choices, 20%] Consider the following systems:

$$
\begin{aligned}
\text{System 1: } y(t) &= x(t-2) + x(2-t) \\
\text{System 2: } y(t) &= x(\sin(t)) \\
\text{System 3: } y[n] &= nx[n] \\
\text{System 4: } y[n] &= \begin{cases} 0, & x[n] < 0 \\ x[n] + x[n-2], & x[n] \geq 0 \end{cases}
\end{aligned}
$$

1. [Outcome 1, 4%] For Systems 1 to 4, determine whether the systems are memoryless.
2. [Outcome 1, 4%] For Systems 1 to 4, determine whether the systems are invertible.
3. [Outcome 1, 4%] For Systems 1 to 4, determine whether the systems are causal.
4. [Outcome 1, 4%] For Systems 1 to 4, determine whether the systems are linear.
5. [Outcome 1, 4%] For Systems 1 to 4, determine whether the systems are time-invariant.

	M	I	C	L	TI
Sys 1	With Memory	Not Invertible	Non-causal	linear	Time-Varying
Sys 2	With Memory	Not Inverible	Non-causal	Linear	Time-Varying
Sys 3	Memoryless	Invertible	Causal	linear	Time-Varying
Sys 4	W. Memory	Not invertible	Causal	Nonlinear	Time-Invariant

Question 3: (15%) A system $x(t) \longrightarrow y(t)$ is linear. The behavior of this system is described as follows. If the input signal $x(t)$ is an even signal, then the system outputs $y(t) = 3x(t)$. If the input signal $x(t)$ is an odd signal, the system outputs $y(t) = x(2t-1)$. Answer the following questions.

1. [Outcome 1, 3%] What do we mean by "even signals"? You can use words to define an even signal, or you can use mathematical expression.
2. [Outcomes 1 and 2, 10%] If we use $x_1(t) = \cos(t) + \sin(2t)$ as the input signal, what is the output $y_1(t)$?
3. [Outcome 1, 2%] Is the system time-invariant?

1. An even signal is such that the flipped image over $t=0$ axis is itself.
Or equivalently $x(t) = x(-t)$ for all t.

2. $\cos(t)$ is an even signal

$\Rightarrow \cos(t) \longrightarrow 3\cos(t)$

$\sin(2t)$ is an odd signal

$\Rightarrow \sin(2t) \longrightarrow \sin(2(2t-1)) = \sin(4t-2)$

$\Rightarrow x_1(t) \longrightarrow 3\cos(t) + \sin(4t-2) = y_1(t)$ #

3. No: since a shift t_0 of the odd input signal gives a $2t_0$ shift of the output

Question 4: (15%) A practical question. A stock index tracking system is tracking the moving average $y[n]$ of the stock index $x[n]$ over the last three days including the current day. A trader says that when the output of the moving average tracking system is 100 points below the current index, namely when $y[n] < x[n] - 100$, he will purchase 200 shares of Stock P the next day.

Suppose we know that the Dow Jones Industrial Average was 11,000 over the past three days, i.e., when $n = -2, -1, 0$. And the indices of the following 8 days are

Day n	Stock Index $x[n]$
1	11134
2	11214
3	11312
4	11379
5	11623
6	11423
7	11032
8	10989

1. [Outcomes 1 and 2, 7%] Write down the system description $y[n]$ in terms of $x[n]$. What is the impulse response $h[n]$ of this system?

2. [Outcomes 1, 2 and 3, 8%] When is the trader going to place the order of Stock P. Justify your answer.

1. $$y[n] = \frac{1}{3} \sum_{k=0}^{2} x[n-k]$$

$$h[n] = \frac{1}{3} \sum_{k=0}^{2} \delta[n-k]$$

2. $$y[0] = \frac{(11000 + 11000 + 11000)}{3} = 11000 \nless x[0]-100$$

$$y[1] = \frac{11000 + 11000 + 11134}{3} = 11044\tfrac{2}{3} \nless x[1]-100$$

$$y[2] = \frac{11000 + 11134 + 11214}{3} = 11116 \nless x[2]-100$$

$$y[3] = \frac{11134 + 11214 + 11312}{3} = 11220 \nless x[3]-100$$

$$y[4] = \frac{11214 + 11312 + 11379}{3} = 11301\frac{2}{3} \not< \underset{-100}{x[4]}$$

$$y[5] = \frac{11312 + 11379 + 11623}{3} = 11438 < \underset{-100}{x[5]}$$

$\Rightarrow$ The trader will purchase 200 shares of Stock P at Day 6

Question 5: (25%) Consider a system $y[n] = (x[n] + x[n-1])^2 + 0.5y[n-1]$, and assume that the $y[n] = 0$ for all $n < 0$.

1. [Outcome 2, 6%] Find the impulse response $h[n]$.
2. [Outcome 2, 9%] If the input is $x[n] = \mathcal{U}[n] - \mathcal{U}[n-2]$, find the output $y[n]$.
3. [Outcomes 1 and 3, 8%] Compute $x[n] * h[n]$.
4. [Outcome 3, 2%] Is $y[n] = x[n] * h[n]$? Explain your results.

1. Let $x[n] = \delta[n]$

$y[0] = (1+0)^2 + 0.5 \times 0 = 1$

$y[1] = (0+1)^2 + 0.5 \times 1 = 1.5$

$y[2] = (0+0)^2 + 0.5 \times 1.5 = \frac{3}{2} \times \frac{1}{2}$

$y[3] = (0+0)^2 + 0.5 \times \frac{3}{4} = 3 \times (\frac{1}{2})^3$

$\Rightarrow y[n] = \delta[n] + 3 \times (\frac{1}{2})^n u[n-1]$

2. $y[0] = (1+0)^2 + 0.5 \times 0 = 1$

$y[1] = (1+1)^2 + 0.5 \times 1 = 4.5$

$y[2] = (0+1)^2 + 0.5 \times 4.5 = \frac{13}{4}$

$y[3] = (0+0)^2 + 0.5 \times \frac{13}{4} = \frac{13}{8}$

$\Rightarrow y[n] = \delta[n] + \frac{9}{2}\delta[n-1] + 13 \times (\frac{1}{2})^n u[n-2]$

3. $x[n] * y[n]$ can be obtained by viewing it as a linear system

Since $x[n] = \delta[n] + \delta[n-1]$

$\Rightarrow x[n] * h[n]$

$$= \left(\delta[n] + 3*\left(\frac{1}{2}\right)^n u[n-1]\right)$$

$$+ \delta[n-1] + 3*\left(\frac{1}{2}\right)^{n-1} u[n-2]\Big)$$

4. No. Because the system is Not linear.

Question 6: (20%) Consider a LTI system with impulse response $h(t) = \mathcal{U}(t) - \mathcal{U}(t-2)$. Consider an input signal $x(t) = \cos(2\pi t) + \sin(\pi t)$.

1. [Outcomes 1 and 4, 8%] Find out the Fourier series coefficients α_k and the fundamental frequency ω such that $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega t}$.

2. [Outcome 3, 12%] Find the output $y(t)$ by the convolution integral. You may want to use the distributive property of the convolution integral.

1. The period is $L.C.M.\left(\frac{2\pi}{2\pi}, \frac{2\pi}{\pi}\right)$

$= 2$.

$\Rightarrow \omega = \frac{2\pi}{2} = \pi$.

Therefore $x(t)$ is a combination of the 2nd & the 1st harmonic freq components.

Since

$\cos(2\pi t) = \frac{1}{2}\left(e^{j2\omega t} + e^{-j2\omega t}\right)$

& $\sin(\pi t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$

$\Rightarrow \alpha_0 = 0$

$\alpha_1 = \frac{1}{2j}$ $\qquad \alpha_{-1} = -\frac{1}{2j}$

$\alpha_2 = \frac{1}{2}$ $\qquad \alpha_{-2} = \frac{1}{2}$

$\alpha_k = 0$ for $k = \pm 3, \pm 4, \ldots$

2. Let $x_1(t) = \cos(2\pi t)$

$y_1(t) = x_1(t) * h(t)$

$$= h(t) * x_1(t)$$

$$= \int_{-\infty}^{\infty} [u(s) - u(s-2)] \cos(2\pi(t-s))\, ds$$

$$= \int_0^2 \cos(2\pi(t-s))\, ds$$

$$= 0.$$

$$x_2(t) = \sin(\pi t)$$

$$y_2(t) = x_2(t) * h(t)$$

$$= h(t) * x_2(t)$$

$$= \int_{-\infty}^{\infty} [u(s) - u(s-2)] \sin(\pi(t-s))\, ds$$

$$= \int_0^2 \sin(\pi(t-s))\, ds$$

$$= 0.$$

$$\Rightarrow y(t) = y_1(t) + y_2(t) = 0 \quad \text{for all } t.$$