Q1

Shift $x(t)$ by t_0

$$y(t) = \int_{-\infty}^{3t+3} \left|x\left(\tfrac{s}{3} - t_0\right)\right|^2 ds$$

Sub. $t \to t - t_0$

$$y(t) = \int_{-\infty}^{3t-3t_0+3} \left|x\left(\tfrac{s}{3}\right)\right|^2 ds$$

$$\frac{s}{3} = \frac{a}{3} - t_0 \to s = a - 3t_0$$

$a - 3t_0 = 3t - 3t_0 + 3$ Changed bounds.

$$y(t) = \int_{-\infty}^{3t+3} \left|x\left(\tfrac{s}{3} - t_0\right)\right|^2 ds$$

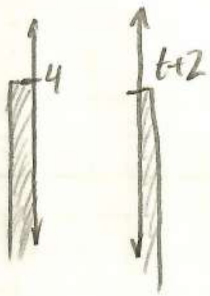
$\boxed{\text{Forms are equal; } \therefore \text{ time invariant.}}$

Q2

$$e^{j3t} = \cos(3t) + j\sin(3t) \rightarrow x(t) = e^{2t-5j-3jt}u(-t+4)$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} \underbrace{u(-\tau+4)e^{2\tau}e^{-3j\tau}e^{-5j}}_{x(\tau)} \underbrace{u(t-\tau+2)e^{-2t}e^{2\tau}}_{h(t-\tau)}$$

Bounds:



Case 1: $t+2 \leq 4$, so $t \leq 2$

Case 2: $t+2 > 4$, so $t > 2$

Case 1:

$$e^{-5j}e^{-2t}\int_{-\infty}^{t+2} e^{4\tau}e^{-3j\tau}\,d\tau = \frac{e^{-5j}e^{-2t}}{4-3j}\left[e^{(4-3j)\tau}\right]_{-\infty}^{t+2}$$

$$= \frac{e^{-5j}e^{-2t}}{4-3j}\left(e^{4t-3jt+8-6j}\right)$$

Case 2:

$$e^{-5j}e^{-2t}\int_{-\infty}^{4} e^{(4-3j)\tau}\,d\tau = \frac{e^{-5j}e^{-2t}}{4-3j}\left[e^{(4-3j)\tau}\right]_{-\infty}^{4}$$

$$= \frac{e^{-5j}e^{-2t}e^{16}e^{-12j}}{4-3j}$$

Positive 2!!!

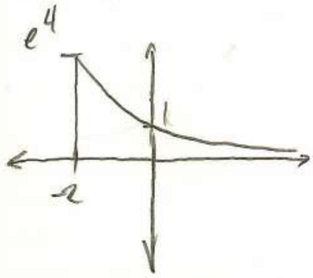
$$\boxed{y(t) = \begin{cases} \dfrac{e^{-11j}e^{2t}e^{-3jt}e^{8}}{4-3j}, & t \leq 2 \\[2ex] \dfrac{e^{-17j}e^{-2t}e^{16}}{4-3j}, & t > 2 \end{cases}}$$

Q2.1 Alternate

An impulse response is the resulting output $y(t)$ when the input $x(t)$ of a LTI system is fed an impulse function $\delta(t)$.

Q2.2 Alternate

Not causal. We graph $h(t)$:



We see $h(t)$ extends left of the y-axis

$\therefore$ $h(t)$ not causal.

Q2.3 Alternate

$$x(t) * h(t) = y(t)$$

Q2.4 Alternate

$$e^{-24j} = \cos(-24) + j\sin(-24)$$

Q3.1

$$w[n] = x[n] * y[n]$$

$$= \sum_{k=-\infty}^{\infty} e^{-2k} u[k-1]\, e^{-\sqrt{2\pi} jn} e^{\sqrt{2\pi} jk}$$

$$= e^{-jn\sqrt{2\pi}} \sum_{k=1}^{\infty} e^{-2k} e^{\sqrt{2\pi} jk}$$

Needs a k-1 in exp. $\quad a = 1$, $r = e^{-2+j\sqrt{2\pi}}$

Mult by: $\dfrac{e^{-2} e^{\sqrt{2\pi} j}}{e^{-2} e^{\sqrt{2\pi} j}}$

$$= e^{-2+j\sqrt{2\pi}} e^{-jn\sqrt{2\pi}} \cdot \frac{1}{1-e^{-2+j\sqrt{2\pi}}}$$

$$\boxed{w[n] = \frac{e^{-2+j\sqrt{2\pi}}}{1-e^{-2+j\sqrt{2\pi}}} e^{-jn\sqrt{2\pi}}}$$

Q3.2

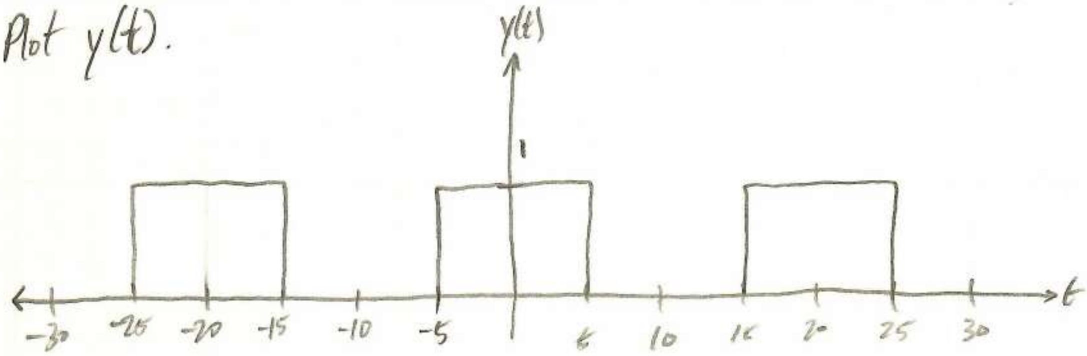
$$\bigstar \quad w[n] = \frac{e^{-2+j\sqrt{2\pi}}}{1-e^{-2+j\sqrt{2\pi}}} x[n]$$

Aha! $\therefore z[n] = y[n] * w[n]$

$$= \frac{e^{-2+j\sqrt{2\pi}}}{1-e^{-2+j\sqrt{2\pi}}} \left(y[n] * x[n] \right)$$

$$\boxed{z[n] = \frac{e^{-4+2j\sqrt{2\pi}}}{(1-e^{-2+j\sqrt{2\pi}})^2} e^{-jn\sqrt{2\pi}}}$$

Q4 Plot y(t).



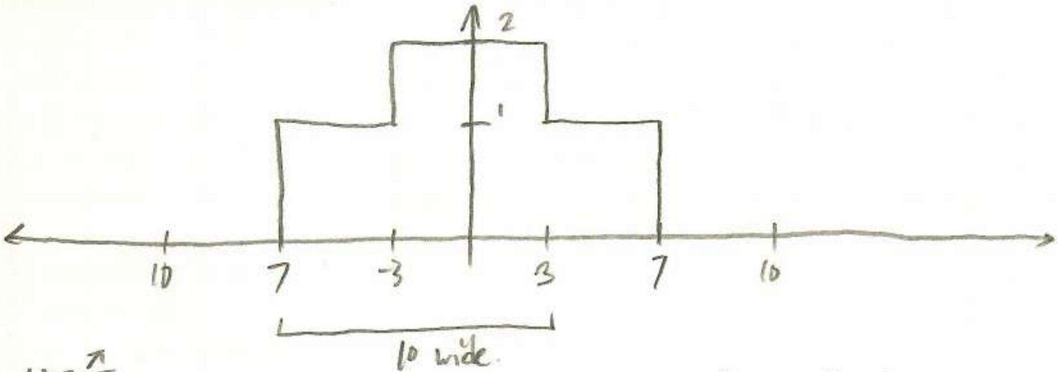
★ Looks like: 1 if $|t| \le 5$

0 if $5 < |t| \le 10$

$T_1 = 5, \ T = 20 \rightarrow \omega_0 = \frac{\pi}{10}$

$$b_7 = \frac{\sin\left(7 \cdot \frac{\pi}{10} \cdot 5\right)}{7\pi} = \frac{\sin\left(\frac{7\pi}{2}\right)}{7\pi} = \boxed{-\frac{1}{7\pi}}$$

Plot w(t)



$\omega_0 = \frac{\pi}{10}$

Looks like $w(t) = y(t-2) + y(t+2)$

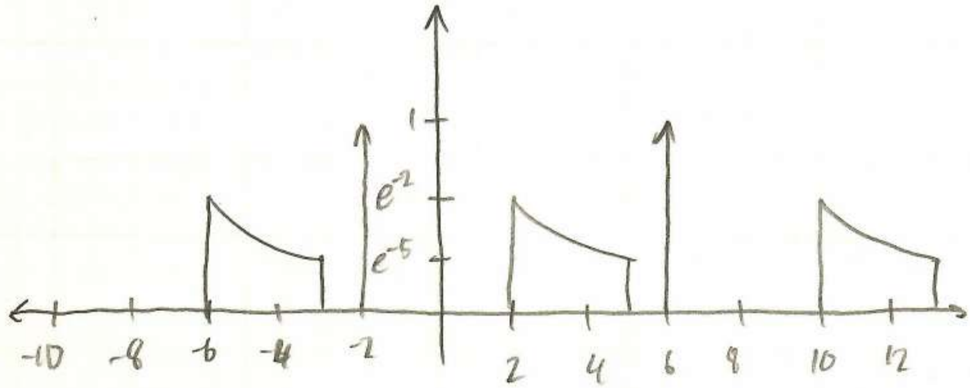
$$\boxed{c_k = b_k e^{-jk\frac{\pi}{5}} + b_k e^{jk\frac{\pi}{5}}}$$

$$c_7 = -\frac{1}{7\pi}\left(e^{j\frac{7\pi}{5}} + e^{-j\frac{7\pi}{5}}\right)$$

$$\boxed{= -\frac{2}{7\pi}\cos\left(\frac{7\pi}{5}\right)}$$

Q5

$x(t)$:



$$a_k = \frac{1}{8}\int_2^{10} x(t)\, e^{-jkt\frac{\pi}{4}}\, dt \qquad \text{as } \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$= \frac{1}{8}\left(\int_2^{5} e^{-t} e^{-jkt\frac{\pi}{4}}\, dt + \int_5^{10} \delta(t-6)\, e^{-jkt\frac{\pi}{4}}\, dt \right)$$

$$= \frac{1}{8}\left(\frac{1}{-1-jk\frac{\pi}{4}} \underbrace{\left[e^{-t-jkt\frac{\pi}{4}} \right]_2^5}_{e^{-5}e^{-jk\frac{5\pi}{4}} - e^{-2}e^{-jk\frac{\pi}{2}}} + \underbrace{e^{-jk\frac{3\pi}{2}}}_{\left(e^{-j\frac{3\pi}{2}}\right)^k = j^k = e^{j\frac{\pi}{2}k}} \right)$$

$$\boxed{a_k = \frac{1}{8}\frac{1}{1+jk\frac{\pi}{4}} e^{-2-jk\frac{\pi}{2}} - \frac{1}{8}\frac{1}{1+jk\frac{\pi}{4}} e^{-5-jk\frac{5\pi}{4}} + \frac{1}{8} e^{jk\frac{\pi}{2}}}$$

Q5-Alt.

$$\sin\left(\frac{\pi}{4}t - \frac{7\pi}{8}\right) \rightarrow T = 8$$

$$e^{jk\omega_0 t} = e^{jk\frac{\pi}{4}t}$$

$$\sin\left(\frac{\pi}{4}t - \frac{7\pi}{8}\right)$$

$$= \frac{1}{2j} e^{-\frac{7\pi}{8}j} e^{j\frac{\pi}{4}t} - \frac{1}{2j} e^{\frac{7\pi}{8}j} e^{-j\frac{\pi}{4}t}$$

Inspection.

$$\boxed{a_1 = \frac{1}{2j} e^{-\frac{7\pi}{8}j} \qquad a_7 = a_{-1} = -\frac{1}{2j} e^{\frac{7\pi}{8}j}}$$

$a_k = 0$ for all other k

Q6

	Yes	No
Causal	○	●
Memoryless	○	●
Stable	●	○
Linear	●	○
Time-Invariant	○	●

System 1

Causality:

$y_1(t) = 2^{-5} x_1(25t) + 2^{-6} x_1(36t) + \ldots$

$t = 1$:

$y_1(1) = 2^{-5} x_1(25) + 2^{-6} x_1(36) + \ldots$

These are future inputs.

∴ Not causal

∴ Not Memoryless

Stability Proof

$|x(t)| < B$: for all t

$|y(t)| = \left| \sum_{k=5}^{\infty} 2^{-k} x_1(k^2 t) \right|$

$|y(t)| < \sum_{k=5}^{\infty} 2^{-k} |x_1(k^2 t)| < B \sum_{k=5}^{\infty} 2^{-k}$

As 2^{-k} converges,

$B \sum_{k=5}^{\infty} 2^{-k}$ converges.

∴ $y_1(t)$ is stable.

L, TI next page.

	Yes	No
Causal	●	○
Memoryless	●	○
Stable	●	○
Linear	○	●
Time-Invariant	●	○

System 2

Of note w/ system 2:

Since $x_2[n+1] < 0$,

$y_2[n] = e^{x_2[n]} \cdot \frac{x_2[n+1]}{|x_2[n+1]|}$

just equals $-e^{x_2[n]}$

∴ $y_2[n] = -e^{x_2[n]}$

Memoryless + Causal:

$y_2[n]$ depends only on current n.

Stability proof: $-B < x_2[n] < B$, $B \neq \infty$

$-e^{-B} > y_2[n] > -e^{B}$

$y_2[n]$ always bounded,

∴ Stable.

L, TI next page.

Linear - Sys. 1

Input $x(t) = \alpha x_a(t) + \beta x_b(t)$

$$= \sum_{k=5}^{\infty} 2^{-k} \left(\alpha x_a(k^2 t) + \beta x_b(k^2 t)\right)$$

$$= \alpha \sum_{k=5}^{\infty} 2^{-k} x_a(k^2 t) + \beta \sum_{k=5}^{\infty} 2^{-k} x_b(k^2 t)$$

Combine outputs $y(t) = \alpha y_a(t) + \beta y_b(t)$

$$= \alpha \sum_{k=5}^{\infty} 2^{-k} x_a(k^2 t) + \beta \sum_{k=5}^{\infty} 2^{-k} x_b(k^2 t)$$

Both are same, so linear

Time - Variance

Shift $x(k^2 t) \rightarrow x(k^2 t - t_0)$

$$y(t) = \sum_{k=5}^{\infty} 2^{-k} x(k^2 t - t_0)$$

Substitute $t \rightarrow t - t_0$

$$y(t-t_0) = \sum_{k=5}^{\infty} 2^{-k} x(k^2 t - k^2 t_0)$$

Not the same. Time Varying.

Linear - Sys. 2

Input $x(n) = \alpha x_a(n) + \beta x_b(n)$

$$= -e^{\alpha x_a[n] + \beta x_b[n]}$$

$$= -e^{\alpha x_a[n]} e^{\beta x_b[n]}$$

Combine Outputs $y[n] = \alpha y_a[n] + \beta y_b[n]$

$$= -\alpha e^{x_a[n]} - \beta e^{x_b[n]}$$

Not same, so not linear

Time - Variance

Shift $x[n] \rightarrow x[n-n_0]$

$$y[n] = -e^{x[n-n_0]}$$

Sub. $n \rightarrow n - n_0$

$$y[n-n_0] = -e^{x[n-n_0]}$$

Same, so time invariant