ECE 301 Solutions | Midterm 1

Q1 - Finding Even/Odd $x[n]$

$$x[-n] = e^{-4|-n-3.25|} = e^{-4|n+3.25|}$$ $\boxed{x[n] \text{ is } \underline{\text{not}} \text{ even.}}$

$x[n]$ re-writable as:

$$\left.\begin{matrix} e^{-4n+13} & n > 3.25 \\ e^{4n-13} & n < 3.25 \end{matrix}\right| \to \text{Discrete: } \begin{matrix} e^{-4n+13} & n \geq 4 \\ e^{4n-13} & n \leq 3 \end{matrix}$$

$Y(z)$ has a natural split:

$$Y(z) = \sum_{n \geq 4}^{\infty} e^{-4n} e^{13} z^{-n} + \sum_{n=-\infty}^{3} e^{4n} e^{-13} z^{-n}$$

We don't <u>have</u> to solve it for z first!

Sub. $\sqrt{2}e^{j\frac{\pi}{4}}$ for z...

$$(e^{-4})^{-n} \left(\sqrt{2}e^{j\pi/4}\right)^{-n}$$

$$Y\left(\sqrt{2}e^{j\pi/4}\right) = \sum_{n=4}^{\infty} (e^{-4})^n \left(\frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}\right)^n e^{13} + \sum_{n=-\infty}^{3} (e^{4})^n \left(\frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}\right)^n e^{-13}$$

Sub. $k-1=n$ Sub. $k-1=-n$

$$\sum_{k=5}^{\infty} (e^{-4})^{k-1} \left(\frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}\right)^{k-1} e^{13} + \sum_{k=-2}^{\infty} (e^{-4})^{k-1} \left(\sqrt{2} e^{j\frac{\pi}{4}}\right)^{k-1} e^{-13}$$

Correct First.

$a = e^{13}$	$a = e^{-13}$
$r = e^{-4} \cdot \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$	$r = e^{-4} \cdot \sqrt{2} e^{j\pi/4}$
$L = 5$	$L = -2$
$\dfrac{e^{13}\left(\frac{1}{\sqrt{2}} e^{-4} e^{-j\frac{\pi}{4}}\right)^4}{1 - \frac{1}{\sqrt{2}} e^{-4} e^{-j\pi/4}}$	$\dfrac{e^{-13}\left(\sqrt{2}\, e^{-4} e^{j\pi/4}\right)^{-3}}{1 - \sqrt{2} e^{-4} e^{j\frac{\pi}{4}}}$

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Q1- Cont.

Left: we invert the r^{z-1}

$$\frac{e^{13}\left(\sqrt{2}e^{4}e^{j\pi/4}\right)^{-4}}{1-\frac{1}{\sqrt{2}}e^{-4}e^{-j\frac{\pi}{4}}}\cdot\frac{-\sqrt{2}e^{4}e^{j\pi/4}}{-\sqrt{2}e^{4}e^{j\pi/4}}$$

$$-\frac{e^{13}\left(\sqrt{2}e^{4}e^{j\pi/4}\right)^{-3}}{1-\sqrt{2}e^{4}e^{j\pi/4}}$$

$$=\frac{e^{1}\left(\sqrt{2}e^{j\pi/4}\right)^{-3}}{1-\sqrt{2}e^{4}e^{j\pi/4}}$$

$$=-\frac{e^{1}}{2\sqrt{2}e^{j3\pi/4}-4e^{4}e^{j\pi}}$$

$$=-\frac{e^{1}}{4e^{4}+2\sqrt{2}e^{j\frac{3\pi}{4}}}$$

Right:

$$\frac{e^{-13}\left(\sqrt{2}e^{-4}e^{j\pi/4}\right)^{-3}}{1-\sqrt{2}e^{-4}e^{j\pi/4}}$$

$$\frac{e^{-1}\left(\sqrt{2}e^{j\pi/4}\right)^{-3}}{1-\sqrt{2}e^{-4}e^{j\pi/4}}$$

$$=\frac{e^{-1}}{2\sqrt{2}e^{j\frac{3\pi}{4}}+4e^{-4}}$$

$$\boxed{\frac{e^{-1}}{4e^{-4}+2\sqrt{2}e^{j\frac{3\pi}{4}}}-\frac{e^{1}}{4e^{4}+2\sqrt{2}e^{j3\pi/4}}}$$

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Q1 - Wrong Formula

$a = e^{13}$

$r = e^{-4} \cdot \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$

$L = 5$

$a = e^{-13}$

$r = e^{-4}\sqrt{2}e^{j\frac{\pi}{4}}$

$L = -2$

$$\frac{e^{13}\left(e^{-4}\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right)^5}{1-e^{-4}\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}}\left(\frac{-e^{4}\sqrt{2}e^{j\pi/4}}{-e^{4}\sqrt{2}e^{j\pi/4}}\right)$$

$$\frac{e^{-13}\left(e^{-4}\sqrt{2}e^{j\pi/4}\right)^{-2}}{1-e^{-4}\sqrt{2}e^{j\pi/4}}$$

$$\downarrow$$

$$-\frac{e^{13}\left(e^{-4}\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right)^4}{1-e^{4}\sqrt{2}e^{j\pi/4}}$$

$$\frac{e^{-5}\left(\sqrt{2}e^{j\pi/4}\right)}{1-e^{-4}\sqrt{2}e^{j\pi/4}}$$

$$\boxed{-\frac{e^{-3}\left(\sqrt{2}e^{j\pi/4}\right)^{-4}}{1-e^{4}\sqrt{2}e^{j\pi/4}}+\frac{e^{-5}\left(\sqrt{2}e^{j\pi/4}\right)^{-4}}{1-e^{-4}\sqrt{2}e^{j\pi/4}}}$$

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2. First consider the integral.

$$y(t) = \int_{-\infty}^{\infty} x(s)\, h(t+2s)\, ds$$

Because x(s) nonzero iff $s \geq 4$:

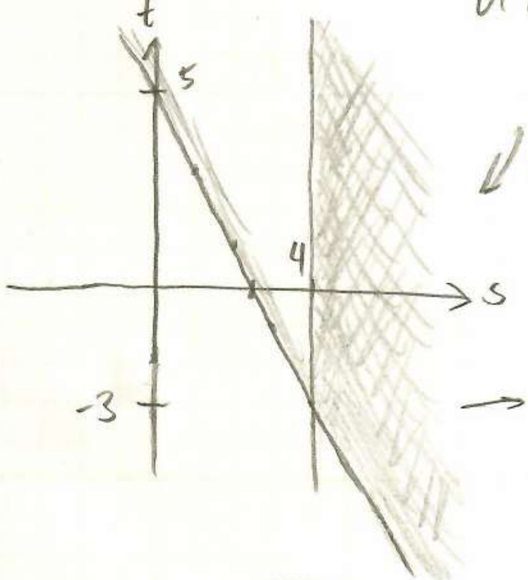
$$= \int_{4}^{\infty} e^{j\frac{\pi}{2}t} \underbrace{h(t+2s)}_{\text{Of interest.}}\, ds$$

$t+2s \geq 5$

t is arbitrary.

$s \geq -\frac{t+5}{2}$

Creates an interesting set of bounds.



We find our solution region!

→ Cross over point: $t=-3$

Lower than that and s must increase above 4 to keep bounds valid.

Case 1: $t \geq -3$

Integral is:

$$\int_4^{\infty} e^{j\frac{\pi}{2}s} e^{-2t} e^{-4s}\, ds$$

@ infinity = 0

$$e^{-2t} \cdot \frac{1}{-4+j\frac{\pi}{2}} \left[e^{(-4+j\frac{\pi}{2})s} \right]_4^{\infty}$$

$$= \frac{e^{-2t}}{-4+j\frac{\pi}{2}} \left[-e^{-16} \cdot \overbrace{e^{-j2\pi}}^{1} \right]$$

$$\boxed{= \frac{e^{-2t-16}}{4-j\pi/2} \text{ when } t \geq -3}$$

Case 2: $t < -3$

Integral is:

$$\int_{-\frac{t+5}{2}}^{\infty} e^{j\frac{\pi}{2}s} e^{-2t} e^{-4s}\, ds$$

$$\frac{e^{-2t}}{-4+j\pi/2} \left[-e^{(-4+j\frac{\pi}{2})\frac{-t+5}{2}} \right]$$

$$= \frac{e^{-2t} \cdot e^{(2t - j\frac{\pi}{4}t - 10 + j\frac{5\pi}{4})}}{4-j\frac{\pi}{2}}$$

$$\boxed{= \frac{e^{-j\frac{\pi}{4}t}\, e^{-10+j\frac{5\pi}{4}}}{4-j\frac{\pi}{2}} \text{ when } t \leq -3}$$

Q3.1

$$x_{odd}(t) = \frac{x(t) - x(-t)}{2}$$

Can we ever get x_{even}? Or just $x(t)$?

$$\boxed{\text{We can't. It's not possible to get } x_{even} \text{ using } x_{odd}.}$$

$$x(t) = x_{odd}(t) + x_{even}(t)$$

- We can't isolate $x(t)$ or $x(-t)$ is the main idea.

Q3.2

$$y(t) = \frac{x(t) + x(-t)}{2}$$

$$x(t) = \begin{cases} 2e^{(1+j)t} & 1 \le t \le 2 \\ 0 & \text{else} \end{cases}$$

$$x(-t) = \begin{cases} 2e^{-(1+j)t} & -2 \le t \le -1 \\ 0 & \text{else} \end{cases}$$

$$y(t) = \begin{cases} e^{(1+j)t} & 1 \le t \le 2 \\ e^{-(1+j)t} & -2 \le t \le -1 \\ 0 & \text{else} \end{cases}$$

$$\int_{-\infty}^{\infty} \|y(t)\|^2 = \int_1^2 \left|e^{(1+j)t}\right|^2 + \int_{-2}^{-1} \left|e^{-(1+j)t}\right|^2$$

$$= \int_1^2 |e^t|^2 \overbrace{|\cancel{e^{jt}}|^2}^{1} + \int_{-2}^{1} |e^{-t}|^2 |\cancel{e^{-jt}}|^2$$

$$= \int_1^2 e^{2t} + \int_{-2}^{1} e^{-2t}$$

$$= \frac{1}{2}\left[e^{2t}\right]_1^2 - \frac{1}{2}\left[e^{-2t}\right]_{-2}^{-1}$$

$$= \frac{1}{2}e^4 - \frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{2}e^4$$

$$\boxed{= e^4 - e^2}$$

Q3.2 Alternative Question

$$w(t) = \begin{cases} 3t\left(\cos(2t+\pi) + j\sin(2t+\pi)\right) & 1 \le t \le 2 \\ 0 & \text{else} \end{cases}$$

Note: $\cos(x) + j\sin(x) = e^{jx}$

$$w(t) = \begin{cases} 3t\, e^{j(2t+\pi)} & 1 \le t \le 2 \\ 0 & \text{else} \end{cases}$$

$$\int_1^2 |w(t)|^2\, dt = \int_1^2 \left|3t\, e^{j(2t+\pi)}\right|^2 dt = \int_1^2 |3t|^2 \overbrace{\cancel{\left|e^{j(2t+\pi)}\right|^2}}^{1} dt$$

$$= \int_1^2 9t^2\, dt = 3\left[t^3\right]_1^2 = 3[8-1]$$

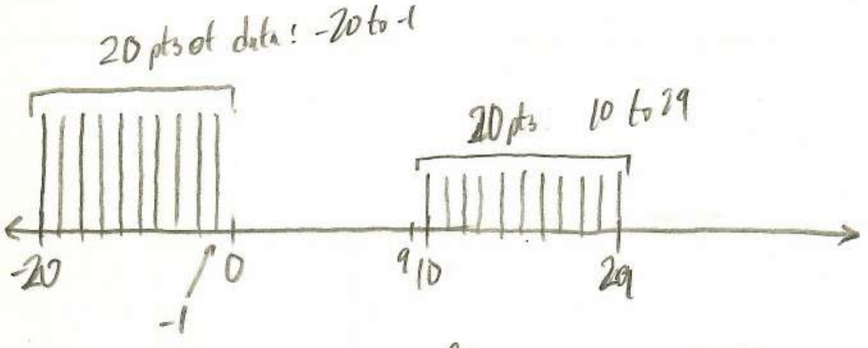
$$\boxed{= 21}$$

Note: $\left|e^{j\theta}\right|$, $\theta \in \mathbb{R}$, always equals 1! Very useful trick!

Absolute values are also commutative over multiplication

or; $|a||b| = |ab|$

Q4. Let's plot it under -20 to 29.



$$\therefore \quad X(k) = \frac{1}{50}\sum_{n=-20}^{-1} 2e^{-jk\frac{2\pi}{50}n} + \frac{1}{50}\sum_{n=10}^{29} e^{-jk\frac{2\pi}{50}n}$$

$$X(10) = \frac{1}{50}\sum_{n=-20}^{-1} 2e^{-jn\frac{2}{5}\pi} + \frac{1}{50}\sum_{n=10}^{29} e^{-jn\frac{2\pi}{5}}$$

$m-1=n$

Substitute.

$$= \frac{1}{50}\left(\sum_{m=-19}^{0} 2e^{-jm\frac{2}{5}\pi} + \sum_{m=11}^{30} e^{-jm\frac{2}{5}\pi}\right)$$

$U-L+1=20$ $a=2$, $U=0$, $L=-19$, $r=e^{-j\frac{2\pi}{5}}$

$a=1$, $U=30$, $L=11$, $r=e^{-\frac{2\pi}{5}j}$ $U-L+1=20$

$$= \frac{2e^{-j\frac{2\pi}{5}\cdot 20}\left(1-e^{-j\frac{2\pi}{5}\cdot 20}\right)}{1-e^{-j\frac{2\pi}{5}}} + \frac{e^{-j\frac{2\pi}{5}\cdot 10}\left(1-e^{-j\frac{2\pi}{5}\cdot 20}\right)}{1-e^{-j\frac{2\pi}{5}}}$$

★ $\left(1-e^{-j\frac{2\pi}{5}20}\right) = 1-e^{-j8\pi} = 1-1 = 0$

$$\boxed{\therefore \text{ Answer} = 0}$$

Q4 - Wrong Formula

$a=2$, $U=0$, $L=-19$, $r=e^{-j\frac{2\pi}{5}}$, $U-L+1=20$

$a=1$, $U=30$, $L=11$, $r=e^{-j\frac{2\pi}{5}}$, $U-L+1=20$

$$\underbrace{\frac{2^{-19}\left(1-e^{-j\frac{2\pi}{5}20}\right)}{1-e^{-j\frac{2\pi}{5}}}}_{\downarrow\, 0} + \underbrace{\frac{1^{-19}\left(1-e^{-j\frac{2\pi}{5}20}\right)}{1-e^{-j\frac{2\pi}{5}}}}_{\downarrow\, 0}$$

$$\boxed{=0}$$

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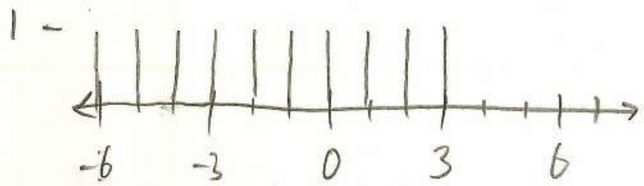
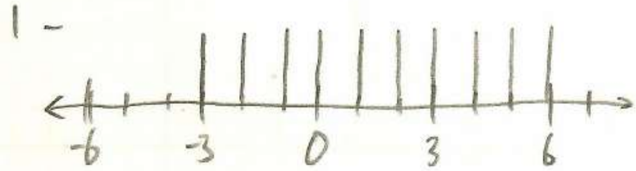
Q5.

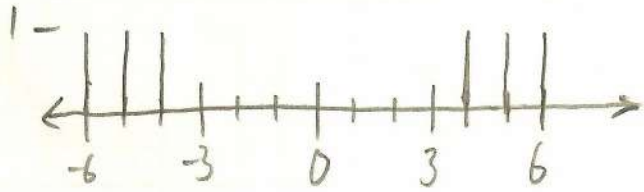
$x_1[n]$



$x_2[n]$



$x[n]$



Q6. As $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$ we simplify $x_1(t)$ to:

$$x_1(t) = -2j\sum_{k=1}^{3} \sin t/k \Rightarrow -2j\sin(t) - 2j\sin(t/2) - 2j\sin(t/3)$$

We see all three have no phase shift, are odd functions, and are inherently periodic.

Term 1: $T = 2\pi$ | Term 2: $T = 4\pi$ | Term 3: $T = 6\pi$ $\rightarrow$ Periodic $\boxed{\text{LCM} \quad T = 12\pi}$

$\boxed{\text{Odd Function}}$

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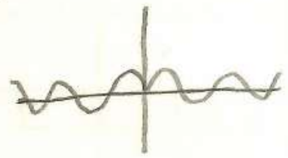
x_2: We have $\int_0^{|t|} \cos 3s \, ds \rightarrow \frac{1}{3}\left[\sin(3|t|) - \sin(0)\right] = \frac{1}{3}\sin(3|t|)$

We are <u>tempted</u> to think $T = \frac{2\pi}{3}$ X No!

$\frac{1}{3}\sin(3|t|)$ looks like:



The middle hump makes it: $\boxed{\text{Not Periodic}}$

It is, however, $\boxed{\text{Even.}}$

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x_3: We have $\int_{n-2}^{n+2} \sin(2s)\,ds \rightarrow$ Solved, $-\frac{1}{2}\cos(2n+4) + \frac{1}{2}\cos(2n-4)$

Let's <u>prove</u> even/odd.

$x_3[-n] = -\frac{1}{2}\cos(-2n+4) + \frac{1}{2}\cos(-2n-4)$

Extract negative since cos is even...

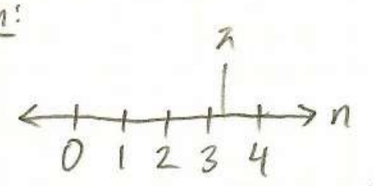
$= -\frac{1}{2}\cos(2n-4) + \frac{1}{2}\cos(2n+4)$

This is $-x_3[n]$, so therefore

$\boxed{x_3[n] \text{ odd}}$

In CT, this is π-periodic.

<u>Problem:</u>



π doesn't <u>ever</u> line up on a discrete value, and n must be an integer

$\boxed{\therefore \text{Not Periodic}}$

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X_4: We have: $\cos(\pi n) + \cos(\pi n^2) + \cos(\pi n^3) + \sin(\pi n) + \sin(\pi n^2) + \sin(\pi n^3)$

$$\downarrow$$

$$\cos(\pi n_1) + \cos(\pi n_2) + \cos(\pi n_3) + \sin(\pi n_1) + \sin(\pi n_2) + \sin(\pi n_3)$$

To show case something:

$n \to n_1$

$n^2 \to n_2$

$n^3 \to n_3$

We perform these substitutions.

n_1, n_2, n_3 <u>must</u> be integers.

Since n_1, n_2, n_3 <u>must</u> be integers,

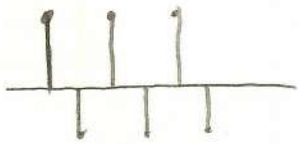
$\sin(\pi n_i) = 0$ for <u>any</u> integer n_i

$\cos(\pi n_i) = (-1)^{n_i}$ ☆ Aha!

$$\therefore X_4 = (-1)^n + (-1)^{n^2} + (-1)^{n^3}$$

$$= (-1)^n + ((-1)^2)^n + ((-1)^3)^n$$

$$= 1 + 2(-1)^n \rightarrow \boxed{\text{which } \underline{\text{is}} \text{ periodic! } T = 2}$$



Proving even/odd: $X_4[-n] = 1 + 2(-1)^{-n} = 1 + 2(-1^{-1})^n = 1 + 2(-1)^n$

or $\frac{1}{-1} = -1$

$\boxed{\text{Even}}$