ECE 301 SP2024 HW7 Solution

Question 61

3.23)

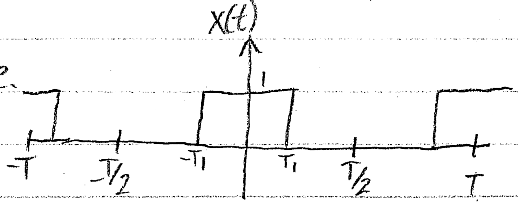
a) $a_k = \begin{cases} 0 & , k=0 \\ (j)^k \dfrac{\sin(\frac{\pi}{4}k)}{\pi k} & , \text{otherwise} \end{cases}$

consider textbook example 3.5

here, a periodic square wave is defined over one period as

$$x(t) = \begin{cases} 1 & , |t| < T_1 \\ 0 & , T_1 \leq |t| \leq T/2 \end{cases}$$

i.e.



for this signal, the Fourier Series coefficients are

$$a_0 = \frac{2T_1}{T} \quad , \quad a_k = \frac{\sin\left(k\frac{2\pi}{T}T_1\right)}{k\pi}$$

Now consider the time shifting property in textbook Table 3.1

$$\text{if} \quad x(t) \xrightarrow{F.S.} a_k$$
$$\text{then} \quad x(t-t_0) \xrightarrow{F.S.} a_k e^{-jk\frac{2\pi}{T}t_0}$$

$$(j)^k \frac{\sin\left(\frac{\pi}{4}k\right)}{\pi k} = \left(\cos\left(\tfrac{\pi}{2}\right)+j\sin\left(\tfrac{\pi}{2}\right)\right)^k \frac{\sin\left(k\frac{2\pi}{4}\cdot\frac{1}{2}\right)}{k\pi}$$

$$= \frac{e^{j\frac{\pi}{2}k}\sin\left(k\cdot\frac{2\pi}{4}\cdot\frac{1}{2}\right)}{k\pi} = e^{-jk\frac{2\pi}{4}(-1)}\frac{\sin\left(k\cdot\frac{2\pi}{4}\cdot\frac{1}{2}\right)}{k\pi}$$

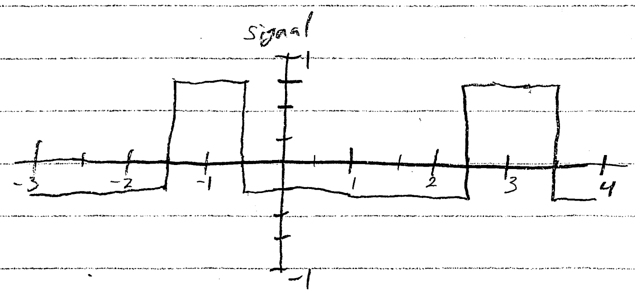
so $t_0 = -1$, $T = 4$, $T_1 = \frac{1}{2}$

$$x(t+1) = \begin{cases} 1 & , |t+1| < \frac{1}{2} \\ 0 & , \frac{1}{2} \le |t+1| \le 2 \end{cases}$$

but, for this $x(t+1)$, $a_0 = \frac{2T_1}{T} = \frac{1}{4}$

so we must subtract $\frac{1}{4}$ from the signal to get $a_0 = 0$

$$\Rightarrow \text{signal} = \begin{cases} 3/4 & , -1.5 < t < -.5 \\ -1/4 & , -3 \le t \le -1.5 \text{ and } -.5 \le t \le 1 \end{cases}$$



b) $a_k = (-1)^k \dfrac{\sin(k\frac{\pi}{8})}{2\pi k}$, $a_0 = \frac{1}{16}$

consider part (A)

$$a_k = (\cos(\pi)+j\sin(\pi))^k \frac{\sin(k\frac{2\pi}{4}\cdot\frac{1}{4})}{2\pi k} = e^{j\pi k}\frac{\sin(k\frac{2\pi}{4}\cdot\frac{1}{4})}{2\pi k}$$

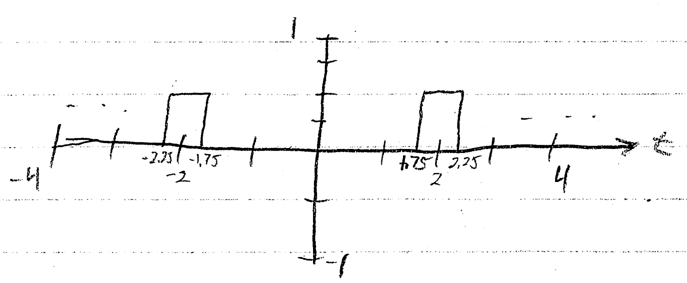
$$a_k = \frac{1}{2} e^{-j\frac{2\pi}{4}k(-2)} \frac{\sin(k\cdot\frac{2\pi}{4}\cdot\frac{1}{4})}{\pi k}$$

$t_0 = -2$, $T = 4$, $T_1 = \frac{1}{4}$

$$\text{signal} = \begin{cases} \frac{1}{2}, & |t+2| < \frac{1}{4} \\ 0, & \frac{1}{4} \le |t+2| \le 2 \end{cases}$$

check $a_0 = \frac{1}{T}\int_{-T_1}^{T_1} \frac{1}{2}\,dt = \frac{T_1}{T} = \frac{\frac{1}{4}}{4} = \frac{1}{16}$ ⇒ good

$$\text{Signal} = \begin{cases} \frac{1}{2}, & -2.25 < t < -1.75 \\ 0, & -4 \le t \le -2.25 \text{ and } -1.75 \le t \le 0 \end{cases}$$



Question 62

3.23)

c.) $a_k = \begin{cases} j^k, & |k| < 3 \\ 0, & \text{else} \end{cases}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} = -2j e^{-j\pi t} - j e^{-j\frac{\pi}{2}t} + j e^{j\frac{\pi}{2}t} + 2j e^{j\pi t}$$

$$x(t) = -\frac{1}{j}\left(2e^{j\pi t} - 2e^{-j\pi t} + e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}\right) \cdot \frac{2}{2}$$

recall $\sin(\theta) = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$\boxed{x(t) = -4\sin(\pi t) - 2\sin(\tfrac{\pi}{2}t)}$$

d.) $a_k = \begin{cases} 1, & k \text{ even} \\ 2, & k \text{ odd} \end{cases}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

seperate the even and odd terms

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{e^{j(2k)\frac{\pi}{2}t}}_{\text{even terms}} + \underbrace{2e^{j(2k+1)\frac{\pi}{2}t}}_{\text{odd terms}} = \left(1 + 2e^{j\frac{\pi}{2}t}\right) \sum_{k=-\infty}^{\infty} e^{j\pi k t}$$

$$\sum_{k=-\infty}^{\infty} (1) e^{jk\frac{2\pi}{2}t} \sim \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

The sum is a F.S. representation of signal with period 4 and F.S. coefficients $= 1$

Now consider textbook example 3.8. Here

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

and all the F.S. coefficients are $a_k = \frac{1}{T}$, so we have

$$\sum_{k=-\infty}^{\infty} (1) e^{jk\frac{2\pi}{2}t} = 2\sum_{k=-\infty}^{\infty} \delta(t-2k) \times 2$$

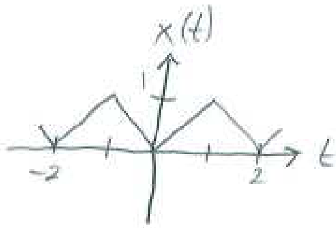
$$x(t) = \left(1 + 2e^{j\frac{\pi}{2}t}\right) 2\sum_{k=-\infty}^{\infty} \delta(t-2k) \times 2$$

$$\boxed{x(t) = \left(2 + 4e^{j\frac{\pi}{2}t}\right) \sum_{k=-\infty}^{\infty} \delta(t-2k)} \times 2$$

Question 63

3.24)

a.) $x(t) = \begin{cases} t & , 0 \le t \le 1 \\ 2-t & , 1 \le t \le 2 \end{cases}$



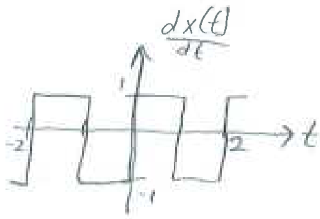
$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$a_0 = \frac{1}{2}\int_0^1 t\,dt + \frac{1}{2}\int_1^2 2-t\,dt$$

$$a_0 = \frac{1}{4}t^2\Big]_0^1 + t - \frac{1}{4}t^2\Big]_1^2 = \left(\frac{1}{4}\right) + \left([2-1] - \left[1-\frac{1}{4}\right]\right)$$

$$\boxed{a_0 = \frac{1}{2}}$$

b.) $\frac{dx(t)}{dt} = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & , 1 < t < 2 \end{cases}$



$$a_0 = \frac{1}{2}\int_0^2 \frac{dx(t)}{dt}\,dt = 0 \qquad \boxed{a_0 = 0}$$

$$a_K = \frac{1}{T}\int_T \frac{dx(t)}{dt} e^{-jK\frac{2\pi}{T}t}dt = \frac{1}{2}\int_0^1 e^{-jK\pi t}dt - \frac{1}{2}\int_1^2 e^{-jK\pi t}dt$$

$$a_K = \frac{-e^{-jK\pi t}}{2jK\pi}\Big]_0^1 + \frac{e^{-jK\pi t}}{2jK\pi}\Big]_1^2$$

$$a_K = \left(\frac{-e^{-jK\pi}}{2jK\pi} - \frac{-1}{2jK\pi}\right) + \left(\cancel{\frac{e^{-jK2\pi}}{2jK\pi}}^{\,1} - \frac{e^{-jK\pi}}{2jK\pi}\right)$$

$$\boxed{a_K = \frac{1}{jK\pi}\left(1 - e^{-jK\pi}\right)}$$

c.) differentiation property

$$x(t) \xrightarrow{FS} a_K \quad \text{then} \quad \frac{dx(t)}{dt} \xrightarrow{FS.} jK\frac{2\pi}{T}a_K$$

Since $\frac{dx(t)}{dt} \xrightarrow{FS} \frac{1}{jK\pi}\left(1-e^{-jK\pi}\right) = a_K$

then $x(t) \xrightarrow{FS} \frac{a_K}{jK\frac{2\pi}{T}} = \frac{\frac{1}{jK\pi}\left(1-e^{-jK\pi}\right)}{jK\frac{2\pi}{2}}$

$$= \frac{1}{j^2K^2\pi^2}\left(1-e^{-jK\pi}\right)$$

So for $x(t)$, the F.S. coefficients are

$$\boxed{a_K = \begin{cases} \frac{1}{2}, & K=0 \\ \frac{e^{-jK\pi}-1}{K^2\pi^2}, & K \neq 0 \end{cases}}$$

Question 64

3.25)

a.) $x(t) = \cos(4\pi t)$, $T = \frac{1}{2}$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt = 2\int_0^{\frac{1}{2}} \cos(4\pi t)\,dt = \frac{2\sin(4\pi t)}{4\pi}\Big]_0^{\frac{1}{2}}$$

$$a_0 = 0$$

$$a_K = 2\int_0^{\frac{1}{2}} \cos(4\pi t)\,e^{-jK4\pi t}\,dt = 2\int_0^{\frac{1}{2}}\left(\frac{1}{2}e^{j4\pi t}+\frac{1}{2}e^{-j4\pi t}\right)e^{-jK4\pi t}\,dt$$

$$a_K = \int_0^{\frac{1}{2}} e^{j4\pi t(1-K)} + e^{-j4\pi t(K-1)}\,dt$$

if $K = \pm 1$, the argument of the integral is one
if $K \neq \pm 1$, then the argument is zero. Can check by
using Euler's Formula $e^{j\theta} = \cos\theta + j\sin\theta$

so...

$$a_K = \begin{cases} 0 & , K \neq \pm 1 \\ \int_0^{\frac{1}{2}} 1 \, dt & , K = \pm 1 \end{cases}$$

$$\boxed{a_K = \begin{cases} 0 & , \text{else} \\ \frac{1}{2} & , K = \pm 1 \end{cases}}$$

b.) $y(t) = \sin(4\pi t)$

$$b_0 = 2\int_0^{\frac{1}{2}} \sin(4\pi t)\, dt = \left.\frac{-2\cos(4\pi t)}{4\pi}\right]_0^{\frac{1}{2}} = 0$$

$$b_K = 2\int_0^{\frac{1}{2}} \sin(4\pi t)\, e^{-jK\frac{2\pi}{1/2}t} dt = \int_0^{\frac{1}{2}} \left(\frac{1}{j}e^{j4\pi t} - \frac{1}{j}e^{-j4\pi t}\right)e^{-jK4\pi t} dt$$

$$= \frac{1}{j}\int_0^{\frac{1}{2}} e^{j4\pi t(1-K)} - e^{j4\pi t(K-1)} dt$$

if $K = 1$, $= \frac{1}{j}\int_0^{\frac{1}{2}} 1 \, dt = \frac{1}{2j}$

if $K = -1$, $= \frac{1}{j}\int_0^{\frac{1}{2}} -1 \, dt = -\frac{1}{2j}$

if $K \neq \pm 1$ $= \frac{1}{j}\int_0^{\frac{1}{2}} 0 \, dt = 0$

$$\boxed{b_K = \begin{cases} 0 & , K \neq \pm 1 \\ \frac{-1}{2j} & , K = -1 \\ \frac{1}{2j} & , K = 1 \end{cases}}$$

c.)

multiplication property

$$x(t) \xrightarrow{FS} a_k \qquad \text{then} \quad x(t)\,y(t) \xrightarrow{FS} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$
$$y(t) \xrightarrow{FS} b_k$$

$$a_l = \frac{1}{2}\delta(l-1) + \frac{1}{2}\delta(l+1)$$

$$b_l = \frac{1}{2j}\delta(l-1) - \frac{1}{2j}\delta(l+1)$$

$$a * b = \frac{1}{2}\delta(l-1) * \left(\frac{1}{2j}\delta(l-1) - \frac{1}{2j}\delta(l+1)\right)$$
$$+ \frac{1}{2}\delta(l+1) * \left(\frac{1}{2j}\delta(l-1) - \frac{1}{2j}\delta(l+1)\right)$$

$$= \frac{1}{4j}\left(\delta(l-2) - \delta(l) + \delta(l) - \delta(l+2)\right)$$

$$= \boxed{\frac{1}{4j}\delta(l-2) - \frac{1}{4j}\delta(l+2)}$$

d.)

$$\sin(4\pi t)\cos(4\pi t) = \frac{1}{2}\sin(8\pi t)$$

$$T = \frac{1}{4}$$

$$a_0 = 4\int_0^{\frac{1}{4}} \frac{1}{2}\sin(8\pi t)\,dt = 0$$

$$a_k = 2\int_0^{\frac{1}{4}} \frac{1}{2j}\left(e^{j8\pi t} - e^{-j8\pi t}\right) e^{-jk8\pi t}\,dt$$

$$a_k = \frac{1}{j}\int_0^{\frac{1}{4}} e^{j8\pi t(1-k)} - e^{j8\pi t(k-1)}\,dt$$

$$\boxed{a_k = \begin{cases} \frac{1}{4j}, & k=1 \\ -\frac{1}{4j}, & k=-1 \\ 0, & \text{else} \end{cases} \qquad \text{Same as part C}}$$

Question 65

3.27)

$$x[n] = \sum a_k e^{jk\frac{2\pi}{N}n}$$

$$x[n] = 2 + 2e^{j\frac{\pi}{6}} e^{j2\frac{2\pi}{5}n} + 2e^{-j\frac{\pi}{6}} e^{j(-2)\frac{2\pi}{5}n} + e^{j\frac{\pi}{3}} e^{j4\frac{2\pi}{5}n} + e^{-j\frac{\pi}{3}} e^{j(-4)\frac{2\pi}{5}n}$$

$$x[n] = 2 + 2e^{j(\frac{4\pi}{5}n + \frac{\pi}{6})} + 2e^{-j(\frac{4\pi}{5}n + \frac{\pi}{6})} + e^{j(\frac{8\pi}{5}n + \frac{\pi}{3})} + e^{-j(\frac{8\pi}{5}n + \frac{\pi}{3})}$$

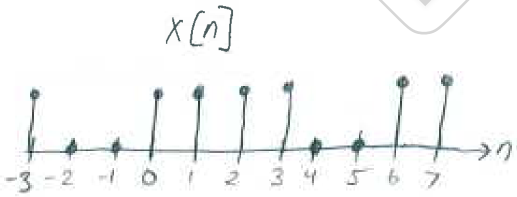
$$x[n] = 2 + 4\cos\left(\frac{4\pi}{5}n + \frac{\pi}{6}\right) + 2\cos\left(\frac{8\pi}{5}n + \frac{\pi}{3}\right)$$

$$\boxed{x[n] = 2 + 4\sin\left(\frac{4\pi}{5}n + \frac{2\pi}{3}\right) + 2\sin\left(\frac{8\pi}{5}n + \frac{5\pi}{6}\right)}$$

Question 66

3.28)

a.)



$N = 6$

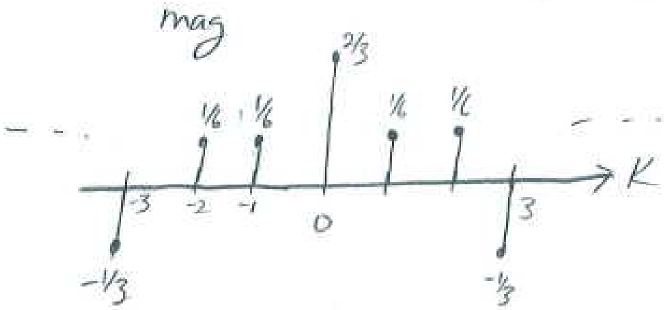
$$a_0 = \frac{1}{6}\sum_{n=0}^{5} x[n] = \frac{4}{6} = \frac{2}{3}$$

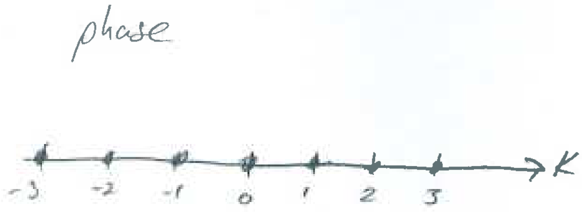
$$a_k = \frac{1}{6}\sum_{n=0}^{5} x[n]\, e^{-jk\frac{\pi}{3}n} = \frac{1}{6}\left(1 + e^{-jk\frac{\pi}{3}} + e^{-jk\frac{2\pi}{3}} + e^{-jk\pi}\right)$$

$$a_k = \frac{1}{6}\left(1 + e^{-jk\frac{\pi}{3}} + e^{jk\frac{\pi}{3}} + e^{-jk\pi}\right)$$

$$\boxed{a_k = \frac{1}{6} + \frac{1}{3}\cos\left(k\frac{\pi}{3}\right) + \frac{1}{6}(-1)^k}$$

Since there are no imaginary parts, the phase is zero





c.) $x[n] = 1 - \sin\frac{\pi n}{4}$ $\quad N=4$

$$a_k = \frac{1}{4}\sum_{n=0}^{3}\left(1-\sin\frac{\pi n}{4}\right)e^{-jk\frac{\pi}{2}n}$$

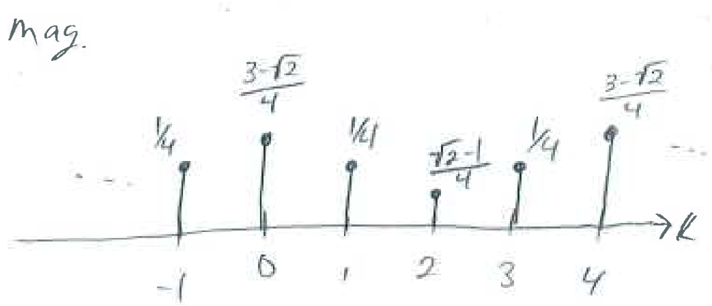
$$a_k = \frac{1}{4}\left(1 + \left(1-\frac{1}{\sqrt{2}}\right)e^{-jk\frac{\pi}{2}} + 0 + \left(1-\frac{1}{\sqrt{2}}\right)e^{-jk\frac{3\pi}{2}}\right)$$

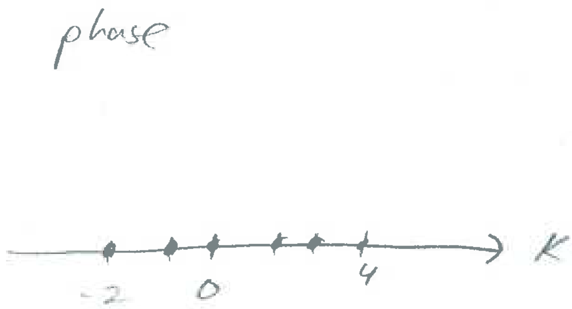
$$a_k = \frac{1}{4}\left(1 + \left(1-\frac{1}{\sqrt{2}}\right)e^{-jk\frac{\pi}{2}} + \left(1-\frac{1}{\sqrt{2}}\right)e^{jk\frac{\pi}{2}}\right)$$

$$a_k = \frac{1}{4}\left(1 + \left(1-\frac{1}{\sqrt{2}}\right)2\cos\left(k\frac{\pi}{2}\right)\right)$$

$$\boxed{a_k = \frac{1}{4} + \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)\cos\left(k\frac{\pi}{2}\right)}$$

a_k is real valued $\Rightarrow$ phase is zero





Question 67

3.29) a.) $a_k = \cos\left(\frac{k\pi}{4}\right) + \sin\left(\frac{3k\pi}{4}\right)$, $N=8$

Consider the signal $x[n] = N(\delta[n-\ell] + \delta[n-(N-\ell)])$ where $1 \le \ell \le N-1$

$$x_1[n] \xrightarrow{FS} a_k = \frac{1}{N}\sum_{n=0}^{N-1}\left(N\delta[n-\ell] + N\delta[n-(N-\ell)]\right) e^{-jk\frac{2\pi}{N}n}$$

$$= e^{jk\frac{2\pi}{N}\ell} + e^{-jk\frac{2\pi}{N}(N-\ell)} = e^{-jk\frac{2\pi}{N}\ell} + e^{-jk2\pi} e^{jk\frac{2\pi}{N}\ell}$$

$$= e^{-jk\frac{2\pi}{N}\ell} + e^{jk\frac{2\pi}{N}\ell} = 2\cos\left(k\frac{2\pi}{N}\ell\right)$$

$$\cos\left(\frac{k\pi}{4}\right) = \frac{1}{2}\left(2\cos\left(k\frac{2\pi}{N}\ell\right)\right)$$

$N=8$ so $\ell = 1$

$$\Rightarrow x_1[n] = \frac{N}{2}\delta[n-\ell] + \frac{N}{2}\delta[n-(N-\ell)] \xrightarrow{FS} \cos\left(\frac{k\pi}{4}\right)$$

$$x_1[n] = 4\delta[n-1] + 4\delta[n-7]$$

Consider the signal $x_2[n] = N\delta[n-\ell] - N\delta[n-(N-\ell)]$

$$x_2[n] \xrightarrow{FS} a_k = \frac{1}{N}\sum_{n=0}^{N-1}\left(N\delta[n-\ell] - N\delta[n-(N-\ell)]\right) e^{-jk\frac{2\pi}{N}n}$$

$$= e^{-jk\frac{2\pi}{N}\ell} - e^{-jk\frac{2\pi}{N}(N-\ell)} = e^{-jk\frac{2\pi}{N}\ell} - e^{-jk2\pi} e^{jk\frac{2\pi}{N}\ell}$$

$$= e^{-jk\frac{2\pi}{N}\ell} - e^{jk\frac{2\pi}{N}\ell} = -2j\sin\left(k\frac{2\pi}{N}\ell\right)$$

$$\sin\left(\frac{3k\pi}{4}\right) = -\frac{1}{2j}\left(-2j\sin\left(k\frac{2\pi}{N}\ell\right)\right)$$

$N=8$ so $\ell = 3$

$$x_2[n] = \frac{-N}{2j}\delta[n-\ell] + \frac{N}{2j}\delta[n-(N-\ell)] \xrightarrow{FS} \sin\left(\frac{3k\pi}{4}\right)$$

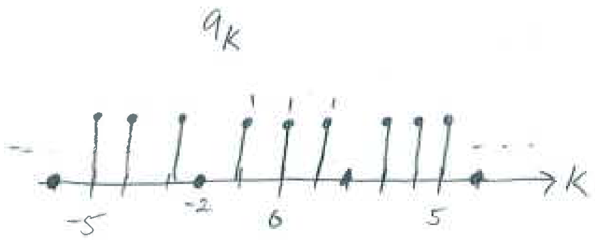
$$x_2[n] = 4j\delta[n-3] - 4j\delta[n-5]$$

Now use the linearity property of Fourier Series to find the signal $x[n]$

$$x[n] = x_1[n] + x_2[n]$$

$$\boxed{x[n] = 4\delta[n-1] + 4\delta[n-7] + 4j\,\delta[n-3] - 4j\,\delta[n-5]}$$

c.)



$N = 8$

$$x[n] = \sum_{n=-2}^{5} a_k e^{jk\frac{\pi}{4}n}$$

$$x[n] = 0 + e^{-j\frac{\pi}{4}n} + 1 + e^{j\frac{\pi}{4}n} + 0 + e^{j\frac{3\pi}{4}n} + e^{j\pi n} + e^{j\frac{5\pi}{4}n}$$

$$x[n] = 1 + 2\cos\left(\tfrac{\pi}{4}n\right) + (-1)^k + e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}$$

$$\boxed{x[n] = 1 + 2\cos\left(\tfrac{\pi}{4}n\right) + (-1)^k + 2\cos\left(\tfrac{3\pi}{4}n\right)}$$

Question 68

3.30)

a.) $x[n] = 1 + \cos\left(\frac{2\pi}{6} n\right)$

$$1 \xrightarrow{FS} \delta[K]$$

$$\cos\left(\frac{2\pi}{6} n\right) \xrightarrow{FS} \frac{1}{2}\left(\delta[K-1] + \delta[K-5]\right)$$

using linearity

$$x[n] \xrightarrow{FS} \delta[K] + \frac{1}{2}\delta[K-1] + \frac{1}{2}\delta[K-5]$$

b.) $y[n] = \sin\left(\frac{2\pi}{6} n + \frac{\pi}{4}\right) = \frac{1}{2j}\left(e^{j\left(\frac{2\pi}{6} n + \frac{\pi}{4}\right)} - e^{-j\left(\frac{2\pi}{6} n + \frac{\pi}{4}\right)}\right)$

$$= \frac{1}{2j} e^{j\frac{2\pi}{6} n} e^{j\frac{\pi}{4}} - \frac{1}{2j} e^{-j\frac{2\pi}{6} n} e^{-j\frac{\pi}{4}}$$

$$= \frac{1}{2j} e^{j\frac{\pi}{4}} e^{j\frac{2\pi}{6} n} - \frac{1}{2j} e^{-j\frac{\pi}{4}} e^{j\frac{10\pi}{6} n}$$

$$= \sum_{K=0}^{5} a_K e^{jK\frac{2\pi}{6} n}$$

where
$$\boxed{\begin{array}{c} a_1 = \dfrac{e^{j\pi/4}}{2j} \\[2ex] a_5 = \dfrac{-e^{-j\frac{\pi}{4}}}{2j} \\[2ex] a_0 = a_2 = a_3 = a_4 = 0 \end{array}}$$

Question 69

$$z[n] = x[n]\,y[n] \leftrightarrow c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

* This is just convolution

$a_k * b_k$:

$$b_1 = \frac{e^{j\pi/4}}{2j} \qquad a_k = \delta[k] + \tfrac{1}{2}\delta[k-1] + \tfrac{1}{2}\delta[k-5]$$

$$b_{-1} = \frac{-e^{-j\pi/4}}{2j}$$

$$b_k = \frac{e^{j\pi/4}}{2j}\delta(k-1) + \frac{e^{-j\pi/4}}{2j}\delta(k+1)$$

$$c_k = \frac{e^{j\pi/4}}{2j}\delta(k-1) - \frac{e^{-j\pi/4}}{2j}\delta(k+1) + \frac{1}{2}\cdot\frac{e^{j\frac{\pi}{4}}}{2j}\delta(k-2) - \frac{1}{2}\cdot\frac{e^{-j\frac{\pi}{4}}}{2j}\delta(k) + \frac{1}{2}\cdot\frac{e^{j\frac{\pi}{4}}}{2j}\underbrace{\delta(k-6)}_{\text{or } \delta(k)} - \frac{1}{2}\cdot\frac{e^{-j\frac{\pi}{4}}}{2j}\underbrace{\delta(k-4)}_{\text{or } \delta(k+2)}$$

$$c_0 = \frac{1}{2}\sin\left(\frac{\pi}{4}\right) \text{ or } \frac{1}{2}\cos\frac{\pi}{4} \qquad c_1 = \frac{e^{j\pi/4}}{2j} \text{ or } \frac{e^{-j\pi/4}}{2} \qquad c_{-1} = -\frac{e^{-j\pi/4}}{2j} \text{ or } \frac{e^{j\frac{\pi}{4}}}{2}$$

$$c_2 = \frac{e^{j\frac{\pi}{4}}}{2j} \text{ or } \frac{e^{-j\pi/4}}{2} \qquad c_{-2} = -\frac{e^{-j\pi/4}}{2j} \text{ or } \frac{e^{j\pi/4}}{2}$$

Question 70

3.33)

a.) given the causal LTI system with impulse response

$$h(t) = e^{-4t}u(t)$$

we know

$$e^{j\omega t} \longrightarrow \boxed{LTI} \longrightarrow e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$$

$x(t) = \cos(2\pi t)$ with period 1

$$= \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}$$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-4\tau}u(\tau)e^{-j\omega\tau}\,d\tau = \int_{0}^{\infty} e^{-(4+j\omega)\tau}\,d\tau$$

$$= \left.\frac{e^{-(4+j\omega)\tau}}{-(4+j\omega)}\right]_0^{\infty} = \frac{1}{4+j\omega}$$

So $$y(t) = \frac{\frac{1}{2}e^{j2\pi t}}{4+j2\pi} + \frac{\frac{1}{2}e^{-j2\pi t}}{4-j2\pi}$$

$$= \frac{1}{8+j4\pi}e^{j2\pi t} + \frac{1}{8-j4\pi}e^{-j2\pi t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{1}t}$$

$$\Rightarrow \boxed{a_k = \begin{cases} 0 & \text{else} \\ \frac{1}{8+j4\pi} & k=1 \\ \frac{1}{8-j4\pi} & k=-1 \end{cases}}$$

6.) $x(t) = \sin 4\pi t + \cos(6\pi t + \pi/4)$

$$x(t) = \frac{1}{2j} e^{j4\pi t} - \frac{1}{2j} e^{-j4\pi t} + \frac{1}{2} e^{j(6\pi t + \pi/4)} + \frac{1}{2} e^{-j(6\pi t + \pi/4)}$$

$$x(t) = \frac{1}{2j} e^{j4\pi t} - \frac{1}{2j} e^{-j4\pi t} + \frac{1}{2} e^{j\frac{\pi}{4}} e^{j6\pi t} + \frac{1}{2} e^{-j\frac{\pi}{4}} e^{-j6\pi t}$$

So

$$y(t) = \frac{\frac{1}{2j} e^{j4\pi t}}{4+j4\pi} - \frac{\frac{1}{2j} e^{-j4\pi t}}{4-j4\pi} + \frac{\frac{1}{2} e^{j\frac{\pi}{4}} e^{j6\pi t}}{4+j6\pi} + \frac{\frac{1}{2} e^{-j\frac{\pi}{4}} e^{-j6\pi t}}{4-j6\pi}$$

$\Rightarrow$

$a_2 = \frac{1}{2j(4+j4\pi)}$	$a_3 = \frac{e^{j\pi/4}}{2(4+j6\pi)}$
$a_{-2} = \frac{-1}{2j(4-j4\pi)}$	$a_{-3} = \frac{e^{-j\pi/4}}{2(4-j6\pi)}$

Question 71

3.34)

recall $\sum_{k=-\infty}^{\infty} \delta(t-kT) \xleftrightarrow{F.S.} a_k = \frac{1}{T} \quad \forall k$ (for all)

$\Rightarrow T\sum_{k=-\infty}^{\infty} \delta(t-kT) \xleftrightarrow{F.S.} a_k = 1 \quad \forall k$

$\updownarrow$ F.S.

$\sum_{k=-\infty}^{\infty} (1) e^{jk\frac{2\pi}{T}t}$

So $T\sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}$

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-4|\tau|} e^{-j\omega\tau} d\tau = \int_{0}^{\infty} e^{-4\tau} e^{-j\omega\tau} d\tau + \int_{-\infty}^{0} e^{4\tau} e^{-j\omega\tau} d\tau$$

$$H(j\omega) = \frac{1}{4+j\omega} + \frac{1}{4-j\omega} = \frac{8}{16+\omega^2}$$

b.) $x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-n)$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-2n) - \sum_{n=-\infty}^{\infty} \delta((t-1)-2n)$$

$\updownarrow$ FS. $\qquad\qquad \updownarrow$ FS

$\frac{1}{2} \ \forall k \qquad\qquad \frac{1}{2} e^{-jk\frac{2\pi}{2}(1)} \ \forall k$

$$\Rightarrow x(t) \xleftrightarrow{F.S.} \frac{1}{2} - \frac{1}{2}e^{-jk\pi} = \frac{1}{2} - \frac{1}{2}(-1)^k \quad \forall k$$

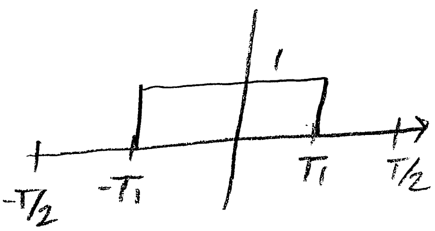
So $x(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2} - \frac{1}{2}(-1)^k\right) e^{jk\frac{2\pi}{2}t}$

$\Rightarrow y(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2} - \frac{1}{2}(-1)^k\right) H\left(jk\frac{2\pi}{2}\right) e^{jk\frac{2\pi}{2}t}$

$$y(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2} - \frac{1}{2}(-1)^k\right) \cdot \frac{8}{16+k^2\pi^2} e^{jk\pi t}$$

$\Rightarrow \boxed{\text{F.S. of } y = \left(\frac{1}{2} - \frac{1}{2}(-1)^k\right) \frac{8}{16+k^2\pi^2} \quad \forall k}$

c.) $x(t)$



$\xleftrightarrow{F.S.} \quad a_k = \begin{cases} \frac{2T_1}{T} & , k=0 \\ \frac{\sin\left(k\frac{2\pi T_1}{T}\right)}{\pi k} & , k \neq 0 \end{cases}$

$T_1 = \frac{1}{4}$

$T = 1$

so $y(t) \xleftrightarrow{F.S.} = \begin{cases} \frac{1}{2} H(j0) & , k=0 \\ \frac{\sin\left(k\frac{\pi}{2}\right)}{\pi k} H(j2\pi k) & , k \neq 0 \end{cases}$

$$= \boxed{\begin{cases} \frac{1}{4} & , k=0 \\ \frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi} \cdot \frac{8}{16+4\pi^2k^2} & , k \neq 0 \end{cases}}$$

Question 72

3.37)

a) $h[n] = \left(\frac{1}{2}\right)^{|n|}$

$$H(j\omega) = \sum_{K=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^{n} e^{-j\omega n} = \sum_{K=1}^{\infty} \left(\tfrac{1}{2} e^{-j\omega}\right)^{n} + \sum_{K=-\infty}^{0} \left(2e^{-j\omega}\right)^{n}$$

$$H(j\omega) = \sum_{K=1}^{\infty} \left(\tfrac{1}{2} e^{-j\omega}\right)^{n} + \sum_{K=0}^{\infty} \left(\frac{1}{2e^{-j\omega}}\right)^{n}$$

$$H(j\omega) = \frac{1}{2e^{j\omega}}\left(\frac{1}{1-\frac{1}{2e^{j\omega}}}\right) + \frac{1}{1-\frac{1}{2e^{-j\omega}}}$$

$$H(j\omega) = \frac{1}{2e^{j\omega}-1} + \frac{2}{2-e^{j\omega}}$$

$x[n] = \sum_{K=-\infty}^{\infty} \delta[n-4K]$ is periodic, $N=4$

$\overset{F.S.}{\longleftrightarrow}$ $a_K = \frac{1}{4}$ for all $K = \langle N \rangle$

$$\Rightarrow x[n] = \sum_{K=\langle N\rangle} \frac{1}{4} H\left(jK\tfrac{2\pi}{4}\right) e^{jK\frac{2\pi}{4}n}$$

So $y[n] \overset{F.S.}{\longleftrightarrow}$

$$\boxed{\begin{array}{l} b_K = \frac{1}{4} H\left(jK\frac{2\pi}{4}\right) \quad \text{for all } K = \langle N \rangle \\ \text{where } H(j\omega) = \frac{1}{2e^{j\omega}-1} + \frac{2}{2-e^{j\omega}} \end{array}}$$

$$6.)\quad x[n] = \begin{cases} 1 & n=0,\pm 1 \\ 0 & n=\pm 2,\pm 3 \end{cases}$$ and is periodic with $N=6$

$$a_k = \frac{1}{6}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi}{6}n}$$

$$a_k = \frac{1}{6}\left(e^{jk\frac{2\pi}{6}} + 1 + e^{-jk\frac{2\pi}{6}}\right)$$

$$a_k = \frac{1}{6}\left(1 + 2\cos\left(k\frac{2\pi}{6}\right)\right)$$

So $y[n] \overset{F.S.}{\longleftrightarrow}$ $b_k = a_k \cdot H\left(jk\frac{2\pi}{6}\right)$ for all $k = \langle 6\rangle$

and

$$H(j\omega) = \frac{1}{2e^{j\omega}-1} + \frac{2}{2-e^{j\omega}}$$

Question 73

3.38)

$$h[n] = \begin{cases} 1 & , 0 \le n \le 2 \\ -1 & , -2 \le n \le -1 \\ 0 & , \text{else} \end{cases}$$

$$H(j\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$= -e^{j\omega 2} - e^{j\omega} + 1 + e^{-j\omega} + e^{-j\omega 2}$$

$$= 1 - 2j\sin(\omega) - 2j\sin(2\omega)$$

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k] = \sum_{k=\langle 4 \rangle} \frac{1}{4} e^{jk\frac{2\pi}{4}n}$$

refer to previous question

$$y[n] = \sum_{k=\langle 4 \rangle} \frac{1}{4} H\left(jk\frac{2\pi}{4}\right) e^{jk\frac{2\pi}{4}n}$$

$$\xleftrightarrow{F.S.} \quad b_k = \frac{1}{4} H\left(jk\frac{2\pi}{4}\right) \quad \forall k = \langle 4 \rangle$$

$$\Rightarrow \quad b_k = \frac{1}{4}\left(1 - 2j\sin\left(k\frac{\pi}{2}\right) - 2j\sin(k\pi)\right)$$

$$= \boxed{\frac{1}{4}\left(1 - 2j\sin\left(k\frac{\pi}{2}\right)\right) \quad \forall k = \langle 4 \rangle}$$

Question 74

3.39)

an input $x[n]$ with period $N=3$ can be represented by

$$x[n] = \sum_{k=\langle 3\rangle} a_k e^{jk\frac{2\pi}{3}n}$$

$$x[n] \xrightarrow[LTI]{h[n]} y[n] = \sum_{k=\langle 3\rangle} a_k H(jk\tfrac{2\pi}{3}) e^{jk\frac{2\pi}{3}n}$$

here $H(j\omega) = \begin{cases} 1 & , |\omega| \le \pi/8 \\ 0 & , \pi/8 \le |\omega| < \pi \end{cases}$

So $y[n] \overset{F.S}{\longleftrightarrow} b_k = a_k H(jk\tfrac{2\pi}{3})$

but $H(jk\tfrac{2\pi}{3}) = \begin{cases} 1 & , k=0 \\ 0 & , k=1,2 \end{cases}$

So $y[n]$ has only one non-zero Fourier Series coefficient.