Question 52

$$h(t) = 3^{-t}u(t) \qquad x(t) = e^{j3t}$$

$$y(t) = \int_{-\infty}^{\infty} 3^{-\tau} e^{j3(t-\tau)} u(\tau)\, d\tau$$

$$= e^{j3t} \int_{0}^{\infty} (3e^{3j})^{-\tau}\, d\tau$$

$$= e^{j3t} \left[\frac{(3e^{3j})^{-\tau}}{-\ln(3e^{3j})} \right]_{\tau=0}^{\infty} = e^{j3t}\left[0 - \frac{1}{-\ln 3 + \ln e^{3j}}\right]$$

$$= \boxed{\frac{e^{j3t}}{\ln(3) + 3j}}$$

$$|y(t)| = \frac{1}{\sqrt{\ln^2(3) + 3^2}} = \boxed{\frac{1}{\sqrt{\ln^2(3) + 9}}}$$

$$\angle y(t) = \angle e^{j3t} - \angle(\ln(3) + 3j)$$

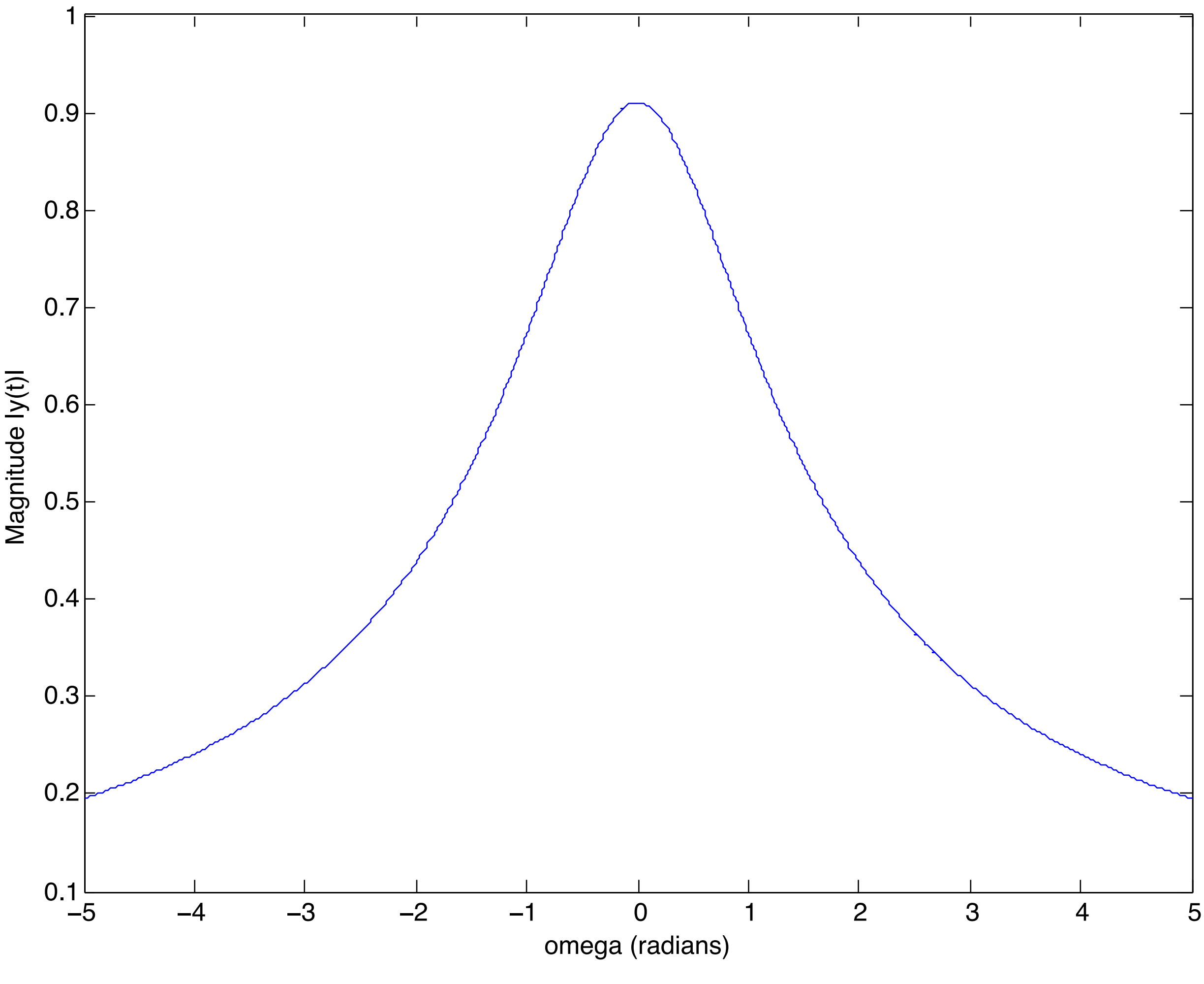
$$= \boxed{3t - \tan^{-1}\left(\frac{3}{\ln(3)}\right)}$$

Question 53

1. $x(t) = e^{j\omega t} \longrightarrow \boxed{y(t) = \frac{e^{j\omega t}}{\ln(3) + j\omega}}$

$$\boxed{|y(t)| = \frac{1}{\sqrt{\ln^2(3) + \omega^2}}}$$

$$\boxed{\angle y(t) = \omega t - \tan^{-1}\left(\frac{\omega}{\ln(3)}\right)}$$


Magnitude |y(t)|
omega (radians)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
−5
−4
−3
−2
−1
0
1
2
3
4
5

Q53-cont

2. The plot on the previous page shows how the magnitude of the system output changes with respect to the frequency of the input to the system when the input is a complex sine wave. This is also known as the magnitude response of the system.

Question 54

$$x(t) = e^{-jt} + 2e^{2jt} + 3e^{j2\sqrt{2}t}$$

$\downarrow$ $\omega = -1$ $\quad$ $\downarrow$ $\omega = 2$ $\quad$ $\downarrow$ $\omega = 2\sqrt{2}$

use y(t) in terms of ω from Q13

$$\boxed{y(t) = \frac{e^{-jt}}{\ln(3) - j} + \frac{2e^{j2t}}{\ln(3) + 2j} + \frac{3e^{j2\sqrt{2}t}}{\ln(3) + 2\sqrt{2}j}}$$

Question 55

3.21) $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

$$= a_0 + \sum_{k=1}^{\infty} a_k e^{jk\frac{2\pi}{T}t} + \sum_{k=-\infty}^{-1} a_k e^{jk\frac{2\pi}{T}t}$$

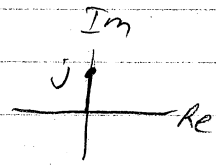
$$= a_0 + \sum_{k=1}^{\infty} a_k e^{jk\frac{2\pi}{T}t} + \sum_{k=1}^{\infty} a_{-k} e^{-jk\frac{2\pi}{T}t}$$

since $x(t)$ is real valued, $a_k = a_{-k}^*$

$$= a_0 + \sum_{k=1}^{\infty} a_k e^{jk\frac{2\pi}{T}t} + \sum_{k=1}^{\infty} a_k^* e^{-jk\frac{2\pi}{T}t}$$

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{a_k e^{jk\frac{2\pi}{T}t}\right\}$$

$$= 0 + 2\,\mathrm{Re}\left\{j e^{j\frac{2\pi}{8}t}\right\} + 2\,\mathrm{Re}\left\{2 e^{j\frac{10\pi}{8}t}\right\}$$



$$= 2\,\mathrm{Re}\left\{e^{j\frac{\pi}{2}} e^{j\frac{\pi}{4}t}\right\} + 2\,\mathrm{Re}\left\{2 e^{j\frac{5\pi}{4}t}\right\}$$

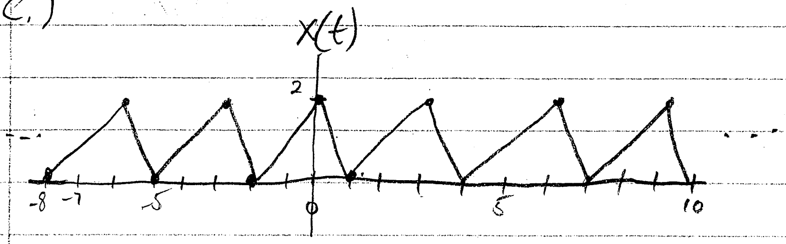
$$= 2\,\mathrm{Re}\left\{e^{j(\frac{\pi}{4}t+\frac{\pi}{2})}\right\} + 4\cos\left(\frac{5\pi}{4}t\right)$$

$$= \boxed{2\cos\left(\frac{\pi}{4}t+\frac{\pi}{2}\right) + 4\cos\left(\frac{5\pi}{4}t\right)}$$

Question 56

3.22) a.)

c.)



$T=3$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jk(\frac{2\pi}{T})t}\,dt$$

$$a_K = \frac{1}{3}\left(\int_{-2}^{0} (t+2)\, e^{-jK(\frac{2\pi}{3})t}\, dt + \int_{0}^{1} (2-2t)\, e^{-jK(\frac{2\pi}{3})t}\, dt\right)$$

let $p = t+2,\ dp = dt$ and $q = 2-2t,\ dq = -2dt$

$t = p-2$ $\quad t = -\frac{1}{2}q+1,\ dt = -\frac{1}{2}dq$

$$a_K = \frac{1}{3}\left(\int_{0}^{2} p e^{-jK(\frac{2\pi}{3})p}\, e^{jK(\frac{4\pi}{3})}\, dp + \int_{2}^{0} q\, e^{jK(\frac{\pi}{3})q}\, e^{-jK(\frac{2\pi}{3})}\left(-\frac{1}{2}\right) dq\right)$$

$$a_K = \frac{1}{3}\left(e^{jK(\frac{4\pi}{3})}\int_{0}^{2} p\, e^{-jK(\frac{2\pi}{3})p}\, dp - \frac{1}{2}e^{-jK(\frac{2\pi}{3})}\int_{2}^{0} q\, e^{jK(\frac{\pi}{3})q}\, dq\right.$$

$\Rightarrow$ integration by parts!

$u = p \quad du = dp$ $\qquad u = q \quad du = dq$

$dv = e^{-jK(\frac{2\pi}{3})p}\, dp$ $\qquad dv = e^{jK(\frac{\pi}{3})q}\, dq$

$V = -\frac{1}{jK(\frac{2\pi}{3})}\, e^{-jK(\frac{2\pi}{3})p}$ $\qquad V = \frac{1}{jK(\frac{\pi}{3})}\, e^{jK(\frac{\pi}{3})q}$

$uv - \int v\, du$

$$a_K = \left[\frac{1}{3}e^{jK(\frac{4\pi}{3})}\right]\left[-\frac{p}{jK(\frac{2\pi}{3})}e^{-jK(\frac{2\pi}{3})p} - \int -\frac{1}{jK(\frac{2\pi}{3})}e^{-jK(\frac{2\pi}{3})p}\, dp\right]$$

$$+\left[-\frac{1}{6}e^{-jK(\frac{2\pi}{3})}\right]\left[\frac{q}{jK(\frac{\pi}{3})}e^{jK(\frac{\pi}{3})q} - \int \frac{1}{jK(\frac{\pi}{3})}e^{jK(\frac{\pi}{3})q}\, dq\right]$$

$$a_K = \left[\frac{1}{3}e^{jK(\frac{4\pi}{3})}\right]\left[-\frac{p}{jK(2\pi/3)}e^{-jK(\frac{2\pi}{3})p} - \frac{1}{j^2K^2(2\pi/3)^2}e^{-jK(2\pi/3)p}\right]_0^2 +$$

$$\left[-\frac{1}{6}e^{-jK(\frac{2\pi}{3})}\right]\left[\frac{q}{jK(\pi/3)}e^{jK(\pi/3)q} - \frac{1}{j^2K^2(\pi/3)^2}e^{jK(\pi/3)q}\right]_2^0$$

$$a_K = \left[\frac{1}{3}e^{jK(4\pi/3)}\right]\left[\left(-\frac{2}{jK(2\pi/3)}e^{-jK(\frac{4\pi}{3})} + \frac{9}{K^2 4\pi^2}e^{-jK(\frac{4\pi}{3})}\right) - \left(0 + \frac{9}{K^2 4\pi^2}\right)\right] +$$

$$\left[-\frac{1}{6}e^{-jK(\frac{2\pi}{3})}\right]\left[\left(0 + \frac{9}{K^2\pi^2}\right) - \left(\frac{2}{jK(\pi/3)}e^{jK\frac{2\pi}{3}} + \frac{9}{K^2\pi^2}e^{jK(\frac{2\pi}{3})}\right)\right]$$

$$a_K = -\frac{1}{jK\pi} + \frac{3}{K^2 4\pi^2} - \frac{3}{K^2 4\pi^2}e^{jK(4\pi/3)} - \frac{6}{K^2\pi^2 4}e^{-jK(\frac{2\pi}{3})} + \frac{1}{j\pi K} + \frac{6}{K^2\pi^2 4}$$

$$a_K = \frac{1}{K^2 4\pi^2}\left(9 - 3e^{jK(4\pi/3)} - 6e^{-jK(2\pi/3)}\right) \quad \text{if } K \neq 0$$

if $K=0$, $\quad a_0 = \frac{1}{3}\left(\int_{-2}^{0} t+2\, dt + \int_{0}^{1} 2-2t\, dt\right)$

$$a_0 = \frac{1}{3}\left(\frac{1}{2}t^2 + 2t \Big]_{-2}^{0} + 2t - t^2 \Big]_0^1 \right)$$

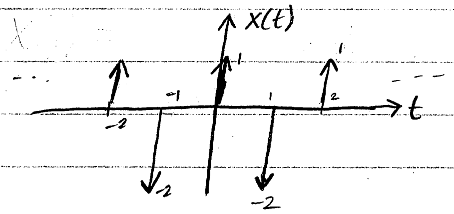
$$a_0 = \frac{1}{3}\left(-(2-4) + (1-0)\right)$$

$$a_0 = \frac{1}{3}(3)$$

$$a_0 = 1$$

$$\boxed{\begin{aligned} a_K &= \begin{cases} 1 & , K=0 \\ \frac{1}{K^2 4\pi^2}\left(9 - 3e^{jK(4\pi/3)} - 6e^{-jK(2\pi/3)}\right) & , K \neq 0 \end{cases} \\ X(t) &= \sum_{K=-\infty}^{\infty} a_K e^{jK\frac{2\pi}{3}t} \end{aligned}}$$

d.)



$T = 2$

$$X(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\frac{2\pi}{T}t} \qquad a_K = \frac{1}{T}\int_T x(t) e^{-jK\left(\frac{2\pi}{T}\right)t} dt$$

$$a_K = \frac{1}{2}\left(e^{-jK\left(\frac{2\pi}{2}\right)0} - 2e^{-jK\left(\frac{2\pi}{2}\right)\cdot 1}\right)$$

$$a_K = \frac{1}{2} - e^{-jK\pi}$$

---

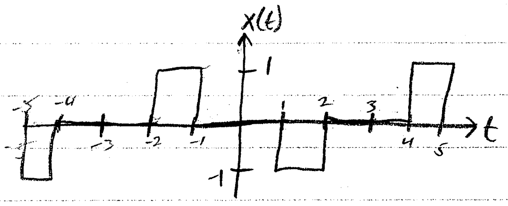
note $e^{-j\pi K} = \cos(-\pi K) + j\sin(-\pi K) = (-1)^n$

$$\boxed{\begin{aligned} a_K &= \frac{1}{2} - (-1)^n \\ X(t) &= \sum_{K=-\infty}^{\infty} a_K e^{jK\pi t} \end{aligned}}$$

Question 57

3.22 a)

e.)



$T=6$

$$a_K = \frac{1}{6}\left(\int_{-2}^{-1} 1\cdot e^{-jK(\frac{2\pi}{6})t}\,dt - \int_{1}^{2} e^{-jK(\frac{2\pi}{6})t}\,dt\right)$$

$$a_K = \frac{1}{6}\left[-\frac{6}{jK2\pi}e^{-jK(\frac{2\pi}{6})t}\right]_{-2}^{-1} - \frac{1}{6}\left[-\frac{6}{jK(2\pi)}e^{-jK(\frac{2\pi}{6})t}\right]_{1}^{2}$$

$$a_K = \frac{1}{6}\left[-\frac{6}{jK2\pi}e^{jK(\frac{2\pi}{6})} + \frac{6}{jK2\pi}e^{jK(\frac{4\pi}{6})}\right] - \frac{1}{6}\left[-\frac{6}{jK2\pi}e^{-jK(\frac{4\pi}{6})} + \frac{6}{jK2\pi}e^{-jK(\frac{2\pi}{6})}\right]$$

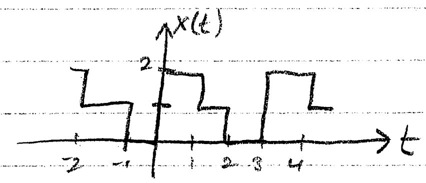
$$a_K = \frac{1}{jK2\pi}\left[-e^{jK\frac{\pi}{3}} + e^{jK\frac{2\pi}{3}} + e^{-jK\frac{2\pi}{3}} - e^{-jK\frac{\pi}{3}}\right]$$

$$a_K = \frac{j}{\pi K}\left[\cos\left(K\frac{\pi}{3}\right) - \cos\left(K\frac{2\pi}{3}\right)\right] \quad \text{for } K\neq 0$$

$$a_0 = \frac{1}{6}\int_{-2}^{4} x(t)\,dt = 0$$

$$\boxed{\begin{gathered} a_K = \begin{cases} 0, & K=0 \\ \frac{j}{\pi K}\left[\cos\left(K\frac{\pi}{3}\right) - \cos\left(K\frac{2\pi}{3}\right)\right], & K\neq 0 \end{cases} \\ x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\frac{2\pi}{6}t} \end{gathered}}$$

f.)



$T=3$

$$a_K = \frac{1}{3}\left(\int_{0}^{1} 2e^{-jK\frac{2\pi}{3}t}\,dt + \int_{1}^{2} e^{-jK\frac{2\pi}{3}t}\,dt\right)$$

$$= \frac{1}{3}\left(\frac{6}{-jK2\pi}e^{-jK\frac{2\pi}{3}t}\Big]_{0}^{1} + \frac{3}{-jK2\pi}e^{-jK\frac{2\pi}{3}t}\Big]_{1}^{2}\right)$$

$$a_k = \frac{1}{3}\left[\frac{6}{jk2\pi} - \frac{6}{jk2\pi} e^{-jk\frac{2\pi}{3}}\right] + \frac{1}{3}\left[\frac{3}{jk2\pi} e^{-jk\frac{2\pi}{3}} - \frac{3}{jk2\pi} e^{-jk\frac{4\pi}{3}}\right]$$

$$a_k = \frac{1}{jk2\pi}\left[2 - e^{-jk\frac{2\pi}{3}} - e^{-jk\frac{4\pi}{3}}\right] \quad , k \neq 0$$

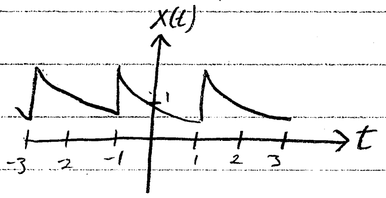
$$a_0 = \frac{1}{3}\int_0^3 x(t)\,dt = \frac{1}{3}(2+1) = 1$$

$$\boxed{\begin{aligned} a_k &= \begin{cases} 1 & , k=0 \\ \frac{1}{jk2\pi}\left[2 - e^{-jk\frac{2\pi}{3}} - e^{-jk\frac{4\pi}{3}}\right] & , k \neq 0 \end{cases} \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{3}t} \end{aligned}}$$

Question 58

3.22)

b)



$T = 2$

$$a_k = \frac{1}{2}\int_{-1}^{1} e^{-t} e^{-jk\left(\frac{2\pi}{2}\right)t}\,dt = \frac{1}{2}\int_{-1}^{1} e^{-t(1+j\pi k)}\,dt$$

$$a_k = \frac{1}{2}\left[-\frac{1}{1+j\pi k} e^{-t(1+j\pi k)}\right]_{-1}^{1}$$

$$a_k = \frac{1}{2}\left[\frac{-1}{1+j\pi k} e^{-1-j\pi k} + \frac{1}{1+j\pi k} e^{1+j\pi k}\right] = \frac{1}{2+j2\pi k}\left[e\, e^{j\pi k} - \frac{1}{e} e^{-j\pi k}\right], \ k \neq 0$$

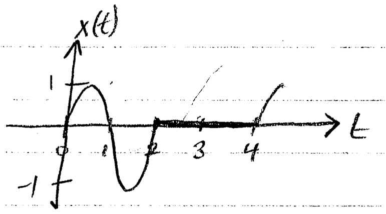
note $e^{j\pi k} = \cos(\pi k) + j\sin(\pi k) = (-1)^n$

$e^{-j\pi k} = \cos(-\pi k) + j\sin(-\pi k) = (-1)^n$

if $k=0$, $a_0 = \frac{1}{2}\int_{-1}^{1} e^{-t}\,dt = \left(\frac{1}{2}\right)\left(-e^{-t}\Big]_{-1}^{1}\right) = \frac{1}{2}\left(e^{1} - e^{-1}\right)$

$$\boxed{\begin{aligned} a_k &= \begin{cases} \frac{1}{2+j2\pi k}\left[(-1)^n\left(e - \frac{1}{e}\right)\right] & , k \neq 0 \\ \frac{1}{2}\left(e - \frac{1}{e}\right) & , k = 0 \end{cases} \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j\pi k t} \end{aligned}}$$

c.)



$T = 4$

$$a_K = \frac{1}{4}\int_0^2 \sin(\pi t) e^{-jK\frac{2\pi}{4}t} dt = \frac{1}{4}\int_0^2 \frac{1}{2j}\left(e^{j\pi t} - e^{-j\pi t}\right) e^{-jK\frac{\pi}{2}t} dt$$

$$a_K = \frac{1}{8j}\int_0^2 e^{j\pi t(1-\frac{K}{2})} - e^{-j\pi t(1+\frac{K}{2})} dt$$

$$a_K = \frac{1}{8j}\left[\frac{1}{j\pi(1-\frac{K}{2})} e^{j\pi t(1-\frac{K}{2})} + \frac{1}{j\pi(1+\frac{K}{2})} e^{-j\pi t(1+\frac{K}{2})}\right]_0^2$$

$$a_K = \frac{1}{8j}\left[\frac{1}{j\pi(1-\frac{K}{2})} e^{j2\pi(1-\frac{K}{2})} + \frac{1}{j\pi(1+\frac{K}{2})} e^{-j2\pi(1+\frac{K}{2})} - \frac{1}{j\pi(1-\frac{K}{2})} - \frac{1}{j\pi(1+K/2)}\right]$$

$$\boxed{\begin{aligned} a_K &= \frac{1}{8}\left[\frac{1}{\pi(1-\frac{K}{2})}\left[1 - e^{j2\pi(1-\frac{K}{2})}\right] + \frac{1}{\pi(1+\frac{K}{2})}\left[1 - e^{-j2\pi(1+\frac{K}{2})}\right]\right] \\ X(t) &= \sum_{K=-\infty}^{\infty} a_K e^{jK\frac{\pi}{2}t} \end{aligned}}$$

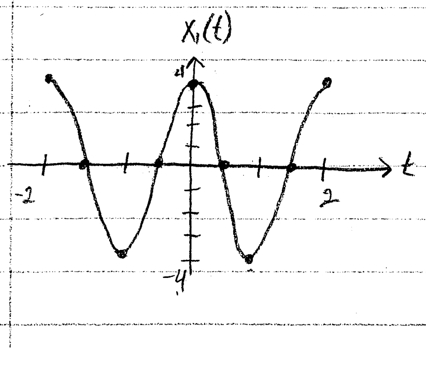
Question 59

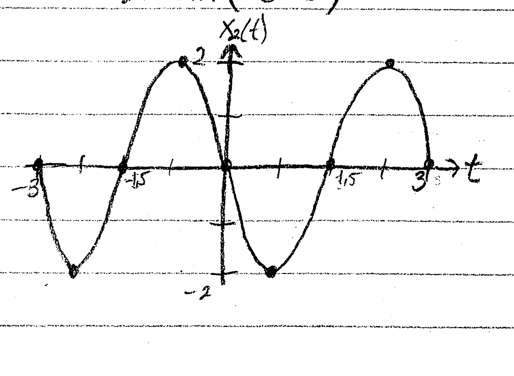
- $x_1(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\frac{2\pi}{T}t}$

$$= (2) e^{j(-1)\frac{2\pi}{2}t} + (2) e^{j(1)\frac{2\pi}{2}t}$$
$$= 2e^{j\pi t} + 2e^{-j\pi t}$$
$$= 4\left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right)$$
$$= 4\cos(\pi t)$$

$X_2(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\frac{2\pi}{T}t}$

$$= (-j) e^{j(-1)\frac{2\pi}{3}t} + (j) e^{j(1)\frac{2\pi}{3}t}$$
$$= \frac{2j}{2}\left(e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}\right)$$
$$= -2\left(\frac{e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}}{2j}\right)$$
$$= -2\sin\left(\frac{2\pi}{3}t\right)$$





- $y(t) = x_1(t) + x_2(t)$

for $y(t)$, $a_{k,y} = \frac{1}{T}\int_T y(t)\, e^{-jk\frac{2\pi}{T}t}\,dt$

$$a_{k,y} = \frac{1}{T}\int_T \left[x_1(t) + x_2(t)\right] e^{-jk\frac{2\pi}{T}t}\,dt$$

$$a_{k,y} = \frac{1}{T}\int_T x_1(t)\, e^{-jk\frac{2\pi}{T}t}\,dt + \frac{1}{T}\int_T x_2(t)\, e^{-jk\frac{2\pi}{T}t}\,dt$$

$a_{k,y} = a_{k,x_1} + a_{k,x_2}$ where a_{k,x_1} and a_{k,x_2} are the Fourier Series coefficients of $x_1(t)$ and $x_2(t)$, respectively.

$$\Rightarrow \boxed{a_{k,y} = \begin{cases} 2-j & , k=-1 \\ 2+j & , k=1 \\ 0 & , k \neq \pm 1 \end{cases} \qquad y(t) = \sum_{k=-\infty}^{\infty} a_{k,y}\, e^{jk\frac{2\pi}{6}t}}$$

Question 60

1.) $\alpha_k = \delta[k-3] + \delta[k+3] \Rightarrow \alpha_k = \begin{cases} 1 & , k = \pm 3 \\ 0 & , k \neq \pm 3 \end{cases}$

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\frac{2\pi}{T}t} = e^{-j\frac{6\pi}{10}t} + e^{j\frac{6\pi}{10}t} = 2\cos\left(\tfrac{3}{5}\pi t\right)$$

$$\boxed{x(t) = 2\cos\left(\tfrac{3\pi}{5}t\right)}$$

2.) $\alpha_k = e^{-2|k|} \Rightarrow \alpha_k = \begin{cases} e^{-2k} & , k \geq 0 \\ e^{2k} & , k < 0 \end{cases}$

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\frac{2\pi}{T}t} = \sum_{k=0}^{\infty} e^{-2k} e^{jk\frac{\pi}{5}t} + \sum_{k=-\infty}^{-1} e^{2k} e^{jk\frac{\pi}{5}t}$$

$$x(t) = \sum_{k=0}^{\infty} \left(e^{j\frac{\pi}{5}t - 2}\right)^k + \sum_{k=-\infty}^{-1} \left(e^{j\frac{\pi}{5}t + 2}\right)^k$$

---

let $l = -k-1 \Rightarrow k = -l-1$

$$X(t) = \sum_{k=0}^{\infty} \left(e^{j\frac{\pi}{5}t - 2}\right)^k + \sum_{\ell=0}^{\infty} \left(e^{j\frac{\pi}{5}t + 2}\right)^{-1} \left(e^{-j\frac{\pi}{5}t - 2}\right)^{\ell}$$

Recall $\sum_{k=0}^{\infty} r(a)^k = \frac{r}{1-a}$

$$\boxed{X(t) = \frac{1}{1-e^{j\frac{\pi}{5}t-2}} + \frac{e^{-j\frac{\pi}{5}t-2}}{1-e^{-j\frac{\pi}{5}t-2}}}$$

3.)

- if $T = \sqrt{3}$, $\boxed{X(t) = 2\cos\left(\frac{6\pi}{\sqrt{3}}t\right) = 2\cos\left(2\pi\sqrt{3}\,t\right)}$
- if $T = \sqrt{3}$

$$\boxed{X(t) = \frac{1}{1-e^{j\frac{2\pi}{\sqrt{3}}t-2}} + \frac{e^{-j\frac{2\pi}{\sqrt{3}}t-2}}{1-e^{-j\frac{2\pi}{\sqrt{3}}t-2}}}$$