Question 43

$\boxed{2.21}$

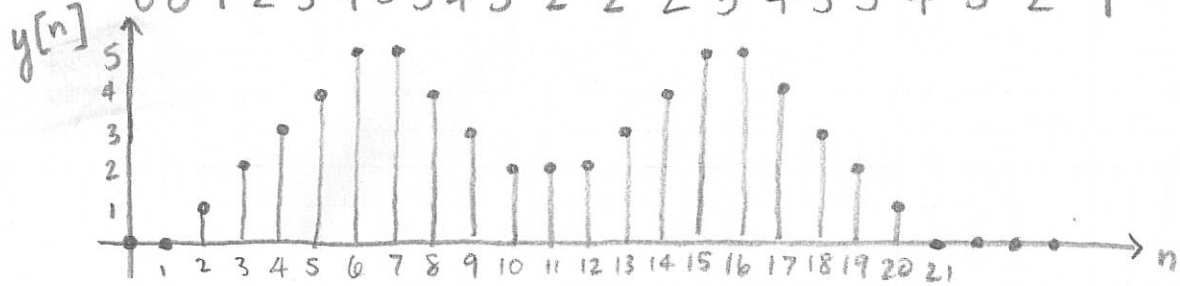
b) $\alpha^n u[n] * \alpha^n u[n]$

$$= \sum_{k=-\infty}^{\infty} \alpha^k \alpha^{n-k} u[k]\, u[n-k]$$

$$= \alpha^n \sum_{k=0}^{n} \alpha^{(k-k)} = \boxed{(n+1)\alpha^n u[n]}$$

d)

		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
k=0	1	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	0	0
k=1	1	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	0
k=2	1	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0
k=3	1	0	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0
k=4	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0
		0	0	1	2	3	4	5	5	4	3	2	2	2	3	4	5	5	4	3	2	1	



Question 44

$\boxed{2.22}$ a) $y(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau} e^{-\beta(t-\tau)}\, d\tau$

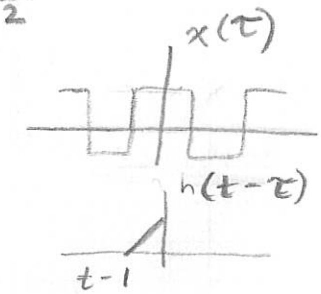
$$= e^{-\beta t} \int_{0}^{t} e^{(\beta-\alpha)\tau}\, d\tau = \frac{e^{-\beta t}}{\beta-\alpha}\left[e^{(\beta-\alpha)\tau} \right]_{\tau=0}^{\tau=t} u(t) = \boxed{\frac{e^{-\beta t}\left(e^{(\beta-\alpha)t} - 1\right)u(t)}{\beta-\alpha} \quad \text{if } \alpha \neq \beta}$$

if $\alpha = \beta \longrightarrow \boxed{y(t) = te^{-\beta t} u(t)}$

Question 44-Continued

e) * result will be periodic, so only determine one period $\rightarrow -\frac{1}{2} < t < \frac{3}{2}$

$$x(\tau) = \begin{cases} 1, & -\frac{1}{2} < \tau < \frac{1}{2} \\ -1, & \frac{1}{2} < \tau < \frac{3}{2} \end{cases}$$



$$h(t-\tau) = \begin{cases} 1-t+\tau, & t-1 \le \tau \le t \\ 0, & \text{else} \end{cases}$$

case 1 $\rightarrow$ $t < -\frac{1}{2} \longrightarrow 0$

case 2 $\rightarrow$ $-\frac{1}{2} < t < \frac{1}{2}$

$$y(t) = \int_{-\frac{1}{2}}^{t} (1-t+\tau)\,d\tau + \int_{t-1}^{-1/2} -(1-t+\tau)\,d\tau$$

$$= \left[\tau - t\tau + \tfrac{1}{2}\tau^2\right]_{\tau=-\frac{1}{2}}^{t} + \left[-\tau + t\tau - \tfrac{1}{2}\tau^2\right]_{\tau=t-1}^{-1/2}$$

$$= t - t^2 + \tfrac{1}{2}t^2 - \left(-\tfrac{1}{2} + \tfrac{1}{2}t + \tfrac{1}{8}\right) + \tfrac{1}{2} - \tfrac{1}{2}t - \tfrac{1}{8} - \left(-t + 1 + t^2 - t - \tfrac{1}{2}t^2 + t - \tfrac{1}{2}\right)$$

$$= -\tfrac{1}{2}t^2 + \tfrac{1}{2}t + \tfrac{3}{8} - \tfrac{1}{8} + \tfrac{1}{2}t - \tfrac{1}{2}t^2$$

$$= \underline{-t^2 + t + \tfrac{1}{4}}$$

case 3 $\longrightarrow$ $\frac{1}{2} < t < \frac{3}{2}$

$$y(t) = \int_{t-1}^{1/2} (1-t+\tau)\,d\tau + \int_{1/2}^{t} -(1-t+\tau)\,d\tau$$

$$= \left[\tau - t\tau + \tfrac{1}{2}\tau^2\right]_{\tau=t-1}^{1/2} + \left[-\tau + t\tau - \tfrac{1}{2}\tau^2\right]_{\tau=1/2}^{t}$$

$$= \tfrac{1}{2} - \tfrac{1}{2}t + \tfrac{1}{8} - t + 1 - (-t^2 + t) - \tfrac{1}{2}t^2 + t - \tfrac{1}{2} - t + t^2 - \tfrac{1}{2}t^2 + \tfrac{1}{2} - \tfrac{1}{2}t + \tfrac{1}{8}$$

$$= \underline{t^2 - 3t + \tfrac{7}{4}}$$

$$\boxed{y(t) = \begin{cases} -t^2 + t + \frac{1}{4}, & -\frac{1}{2} < t \le \frac{1}{2} \\ t^2 - 3t + \frac{7}{4}, & \frac{1}{2} < t < \frac{3}{2} \end{cases} \quad \text{repeated with } T = 2}$$

Question 45

$*$causal if: $h[n] = 0$ for $n < 0$

$*$ stable if: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

remember: $\left(\sum_{n=1}^{\infty} r^n = \frac{1}{1-r} \text{ if } |r|<1 \right)$

$\boxed{2.28}$

b) $h[n] = (0.8)^n u[n+2]$

- not causal
- stable (geometric series with $r<1$)

d) $h[n] = 5^n u[3-n]$

- not causal ($u[3-n]$ turns off at $n=3$)
- stable ($r = \frac{1}{5}$ since n goes to $-\infty$)

f) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[1-n]$

- not causal
- stable ($r = \frac{1}{1.01}$ for negative n)

g) $h[n] = n(\frac{1}{3})^n u[n-1]$

- causal
- stable $\left(\sum_{n=-\infty}^{\infty} h[n] = 1 < \infty \right)$

Question 46

$*$ causal if $h(t) = 0$ for $t < 0$

$*$stable if $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

$\boxed{2.29}$

c) $h(t) = e^{-2t} u(t+50)$

- not causal
- stable

Question 46-Continued

e) $h(t) = e^{-6|t|}$

- not causal
- stable

g) $h(t) = \left(2e^{-t} - e^{\left(\frac{t-100}{100}\right)}\right) u(t)$

- causal
- unstable $\left(\int_{-\infty}^{\infty} |2e^{-t}| dt + \int_{-\infty}^{\infty} e^{\frac{t-100}{100}} dt \right)$

$\downarrow$ finite $\qquad \downarrow$ infinite

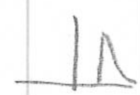
Question 47

$\boxed{2.40}$

a) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2)\, d\tau \qquad \frac{s = \tau - 2}{(\tau = s+2)}$

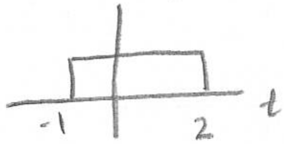
$= \int_{-\infty}^{t-2} x(s)\, e^{-(t-(s+2))} ds = \int_{-\infty}^{\infty} x(s)\, e^{-((t-s)-2)} u((t-s)-2)\, ds$

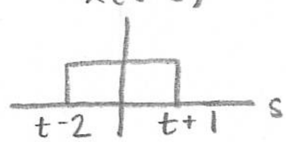
convolution: $\int_{-\infty}^{\infty} x(s) h(t-s)\, ds$ by inspection: $\boxed{h(t) = e^{-(t-2)} u(t-2)}$



$x(t)$ $\qquad$ $x(t-s)$

b)





$-1 < t-s < 2$

$t-2 < s < t+1$

case 1 $\quad t+1 > 2 \;+\; t-2 < 2$

$-1 < t < 4$

$y(t) = \int_{2}^{t+1} e^{-(s-2)} ds$

$= \left[-e^{-(s-2)}\right]_{s=2}^{t+1}$

$= 1 - e^{-(t-1)}$

case 2 $\quad t-2 > 2 \quad t > 4$

$y(t) = \int_{t-2}^{t+1} e^{-(s-2)} ds$

$= \left[-e^{-(s-2)}\right]_{s=t-2}^{s=t+1}$

$= e^{-(t-4)} - e^{-(t-1)}$

$$\boxed{y(t) = \begin{cases} 1 - e^{-(t-1)}, & -1 < t \le 4 \\ e^{-(t-4)} - e^{-(t-1)}, & t > 4 \\ 0, & \text{else} \end{cases}}$$

Question 48

$y[n] = x[n] - 0.5y[n-1] \qquad x[n] = 2^{-n}u[n]$

1. $h[0] = \partial[n] = 1$

$h[1] = \partial[n] - 0.5 = 0 - 0.5 = -\frac{1}{2}$

$h[2] = \partial[n] - (-0.25) = \frac{1}{4}$

$h[3] = \frac{1}{8}$

$h[4] = -\frac{1}{16}$

$$h[n] = (-1)^n \, 2^{-n} u[n] = \boxed{\left(-\frac{1}{2}\right)^n u[n]}$$

2. $y[5] = x[5] * h[5]$

$$= \sum_{n=-\infty}^{\infty} x[5-n]h[n] = \left(-\frac{1}{2}\right)^5 \sum_{n=0}^{5} (-1)^n = 0$$

$$= \frac{1}{2^5} \cdot \left(-\frac{1}{2}\right)^0 + \frac{1}{2^4} \cdot \left(-\frac{1}{2}\right)^1 + \frac{1}{2^3}\left(-\frac{1}{2}\right)^2 + \frac{1}{2^2}\left(-\frac{1}{2}\right)^3$$

$$+ \frac{1}{2^1}\left(-\frac{1}{2}\right)^4 + \frac{1}{2^0}\left(-\frac{1}{2}\right)^5$$

$= 0$

$$\boxed{y[5] = 0}$$

3. $y[n] = \sum_{k=-\infty}^{\infty} (-2)^{-(n-k)} \, 2^{-k} \, u[n]u[n-k]$

$$= \sum_{k=0}^{n} 2^{-k} (-2)^{-n+k} = (-2)^{-n} \sum_{k=0}^{n} (-1)^{-k}$$

$$\boxed{y[n] = \begin{cases} 0, & n < 0 \\ (-2)^{-n}, & \text{even } n \geq 0 \\ 0, & \text{odd } n \geq 0 \end{cases}}$$

Question 49

$$y(t) = \frac{1}{15}\int_{t-15}^{t} x(s)\,ds$$

1. yes, the system is LTI

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

$$y(t) = \frac{1}{15}\int_{t-15}^{t} \alpha_1 x_1(s) + \alpha_2 x_2(s)$$

$$y(t) = \underbrace{\frac{\alpha_1}{15}\int_{t-15}^{t} x_1(s)\,ds}_{y_1(t)} + \underbrace{\frac{\alpha_2}{15}\int_{t-15}^{t} x_2(s)\,ds}_{y_2(t)}$$

$\therefore$ linear

$$x(t+T) \xrightarrow{sys} \frac{1}{15}\int_{t-15}^{t} x(s+T)\,ds$$

let $s' = s+T$

$$\frac{1}{15}\int_{t-15+T}^{t+T} x(s')\,ds' = y(t+T)$$

$$x(t+T) \xrightarrow{sys} y(t+T)$$ $\therefore$ time-invariant

2. $$h(t) = \frac{1}{15}\int_{t-15}^{t} \delta(s)\,ds = \frac{1}{15}\Big[u[s]\Big]_{s=t-15}^{t}$$

$$= \boxed{\frac{1}{15}\Big(u[t] - u[t-15]\Big)}$$

Question 49

3. • not memoryless (has memory)

 • stable

 • causal ($h(t) = 0$ for $t < 0$)

4. $x(t) = \cos(2\pi t)$

$$y(t) = \frac{1}{15}\int_{t-15}^{t} \cos(2\pi s)\, ds = \frac{1}{15\cdot 2\pi}\Big[\sin(2\pi s)\Big]_{s=t-15}^{t}$$

$$y(t) = \frac{1}{30\pi}\sin(2\pi t) - \frac{1}{30\pi}\sin(2\pi(t-15))$$

$*$ since $\sin(2\pi t) = \sin(2\pi t - 30\pi)$,

$$\boxed{y(t) = 0}$$

5. not invertible (if you know the average of a function, there is not a unique mapping back to the original function)

Question 50

$$\dot{y}(t) = -y(t) + x(t)$$

1. $y(t) = e^{-t}u(t) \quad x(t) = \delta(t)$

$$\frac{d}{dt}\left[e^{-t}u(t)\right] = -e^{-t}u(t) + \delta(t)$$

$$\frac{d}{dt}(e^{-t})\cdot u(t) + e^{-t}\cdot\frac{d}{dt}(u(t)) = -e^{-t}u(t) + \delta(t)$$

$$-e^{-t}u(t) + e^{-t}\delta(t) = -e^{-t}u(t) + \delta(t)$$

Question 50-Continued

$*$ the value of e^{-t} at $t=0$ is $e^{0}=1$

so $e^{-t}\delta(t) = \delta t$

$$-e^{-t}u(t) + \delta(t) = -e^{-t}u(t) + \delta(t) \quad \checkmark$$

2. When $x(t) = \delta(t)$, $y(t) = e^{-t}u(t)$

so $\boxed{h(t) = e^{-t}u(t)}$

Question 51

$$x_1 = \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{pmatrix} \quad x_3 = \begin{pmatrix} \sqrt{6}/6 \\ \sqrt{6}/6 \\ -2\sqrt{6}/6 \end{pmatrix}$$

- $|x_1| = |x_2| = |x_3| = 1$

$$x_1 \cdot x_2 = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{3} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{3} + 0 = 0$$

$$x_1 \cdot x_3 = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{6}}{6} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{6}}{6} + 0 = 0$$

$$x_2 \cdot x_3 = \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{6}}{6} + \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{6}}{6} - \frac{\sqrt{3}}{3} \cdot \frac{2\sqrt{6}}{6} = 0$$

$\therefore \{x_1, x_2, x_3\}$ forms an orthonormal basis for $\mathbb{R}^3$

- $x = 0.7\vec{x}_1 + 0.3\vec{x}_2 + 0.4\vec{x}_3$

$$x = \begin{pmatrix} \frac{7\sqrt{2}}{20} + \frac{3\sqrt{3}}{10} + \frac{\sqrt{6}}{15} \\ -\frac{7\sqrt{2}}{20} + \frac{3\sqrt{3}}{10} + \frac{\sqrt{6}}{15} \\ \frac{3\sqrt{3}}{10} - \frac{2\sqrt{6}}{15} \end{pmatrix}$$

Question 51-Continued

$$x' = (7/10,\ 3/10,\ 4/10) = \alpha_1 \vec{X_1}^T + \alpha_2 \vec{X_2}^T + \alpha_3 \vec{X_3}^T$$

since the vectors are orthonormal,

$$\alpha_1 = X' \cdot X_1 \qquad \alpha_2 = X' \cdot X_2 \qquad \alpha_3 = X' \cdot X_3$$

$$\alpha_1 = \frac{7\sqrt{2}}{20} - \frac{3\sqrt{2}}{20} = \boxed{\frac{\sqrt{2}}{5}}$$

$$\alpha_2 = \frac{7\sqrt{3}}{30} + \frac{\sqrt{3}}{10} + \frac{2\sqrt{3}}{15} = \boxed{\frac{7\sqrt{3}}{15}}$$

$$\alpha_3 = \frac{7\sqrt{6}}{60} + \frac{\sqrt{6}}{20} - \frac{8\sqrt{6}}{60} = \boxed{\frac{\sqrt{6}}{30}}$$

- If we can represent any arbitray vector in $\mathbb{R}^3$ as a linear combination of $\{X_1, X_2, X_3\}$, then we can determine the output of any vector for a linear system by taking a linear combination of the outputs for each of the basis vectors, using the same coefficients $\{\alpha_1, \alpha_2, \alpha_3\}$.