ECE 301 Spring 2024 Homework 3 Solution

Question 23

$$x(t) = \underbrace{\frac{x(t)+x(-t)}{2}}_{\text{even}} + \underbrace{\frac{x(t)-x(-t)}{2}}_{\text{odd}}$$

* an even signal satisfies: $x(t) = x(-t)$

$\frac{x(t)+x(-t)}{2} = \frac{1}{2}x(t) + \frac{1}{2}x(-t) \rightarrow$ call this $X_e(t)$

if $X_e(t) = X_e(-t)$, then $X_e(t)$ is an even signal

$X_e(-t) = \frac{1}{2}x(-t) + \frac{1}{2}x(t) = \frac{x(t)+x(-t)}{2}$ ✓ so $X_e(t)$ is even

$$X_o(t) = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$

* an odd signal satisfies $X_o(-t) = -X_o(t)$

$$X_o(-t) = \frac{1}{2}x(-t) - \frac{1}{2}x(t)$$
$$-X_o(t) = -\frac{1}{2}x(t) + \frac{1}{2}x(-t)$$

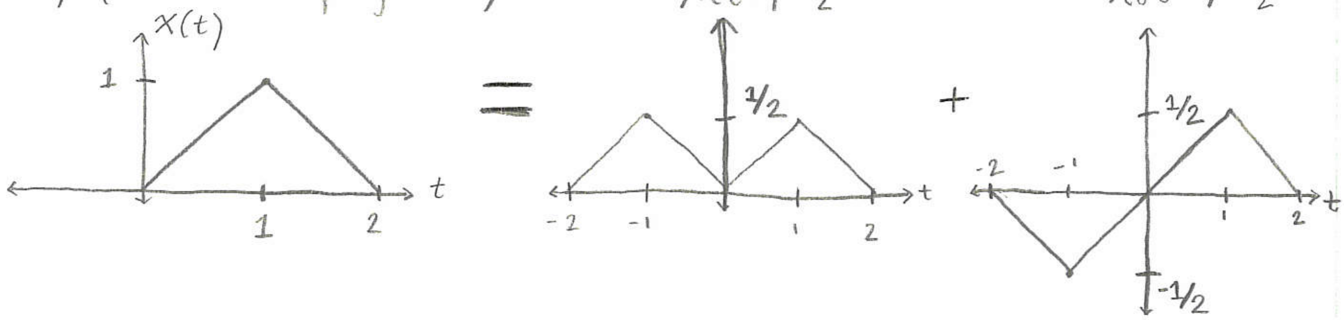
these are equal, so $X_o(t)$ is an odd signal

[This important because it helps prove that any periodic signal can be represented as a weighted sum of sine and cosine signals since a sine signal is odd and a cosine signal is even. This is called the Fourier Series representation of a signal.]

Question 23

a) (1.23a on page 60)

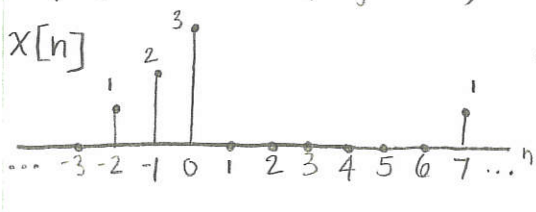
$x_e(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$ $\quad$ $X_o(t) = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$

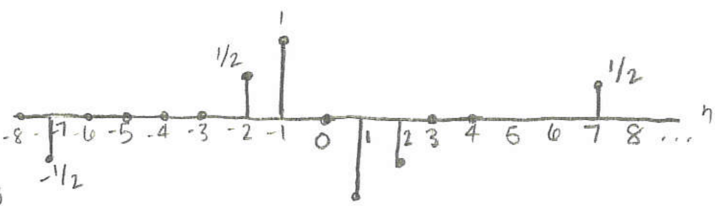


b) (1.24b on page 61)

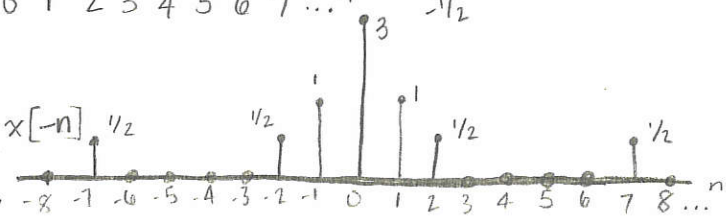
$x_o[n] = \frac{1}{2}x[n] - \frac{1}{2}x[-n]$

x[n]





$x_e[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n]$



Question 24

a) (1.32 on page 63)

$y_1(t) = x(2t)$

$y_2(t) = x\left(\frac{1}{2}t\right)$

(1) True. $y_1(t) = x(2t) = x(2t+T) = y_1\left(t+\frac{T}{2}\right) \rightarrow \boxed{\text{period} = \frac{T}{2}}$

(2) True. $y_1(t) = x(2t) = x(2(t+T)) = x(2t+2T) \rightarrow \boxed{\text{period} = 2T}$

(3) True. $y_2(t) = x(t/2) = x(t/2 + T) = y_2(t+2T) \rightarrow \boxed{\text{period} = 2T}$

(4) True. $y_2(t) = x(t/2) = x\left(\frac{t+T}{2}\right) = y_2\left(t+\frac{T}{2}\right) \rightarrow \boxed{\text{period} = \frac{T}{2}}$

b) (1.33 on page 63-64)

$y_1[n] = x[2n]$

$y_2[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

(1) True.

$x[n] = x[n+N] \rightarrow y_1[n] = x[2n] = x[2n+N] = y_1\left[n+\frac{N}{2}\right]$

$\boxed{\text{period} = \frac{N}{2} \text{ if } N \text{ is even or } N \text{ if } N \text{ is odd}}$

(2) False.

$y_1[n]$ effectively looks at $x[n]$ only for even values of n, or every other value. So if $x[n]$ were something like: $\begin{cases} 1, & n \text{ is even} \\ n, & n \text{ is odd} \end{cases}$ then $y_1[n]$ would always be 1 and periodic even though $x[n]$ is not periodic.

(3) True.

$x[n] = x[n+N]$

$$y_2[n] = \begin{cases} x[\frac{n}{2}], & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases} = \begin{cases} x[\frac{n}{2}+N], & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$

$$= y_2[n+2N] \rightarrow \boxed{\text{period} = 2N}$$

(4) True. ($y[n]$ effectively inserts a sample of 0 between every sample of $x[n]$.)

$y_2[n] = y_2[n+N]$

$$y_2[n] = \begin{cases} x[\frac{n}{2}+N], & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$

$x[n] = y_2[2n]$ for even values of n

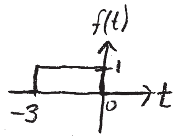
$= y_2[2n+N]$ " "

$= x[n+\frac{N}{2}] \rightarrow \boxed{\text{period} = \frac{N}{2}}$

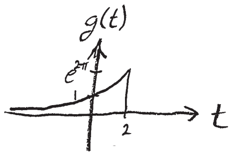
(we know N must be even because of the nature of $y_2[n]$)

Question 25

$$f(t) = \begin{cases} 1 & ; -3 \le t < 0 \\ 0 & ; \text{else} \end{cases}$$

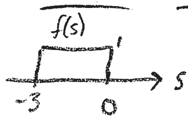


$$g(t) = \begin{cases} e^{\pi t} & , t < 2 \\ 0 & , \text{else} \end{cases}$$

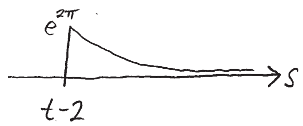


$$h(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\,ds \Rightarrow \text{convolution!}$$

$$f(s) = \begin{cases} 1 & ; -3 \le s < 0 \\ 0 & ; \text{else} \end{cases}$$



$$g(t-s) = \begin{cases} e^{\pi(t-s)} & ; t-s < 2 \\ 0 & ; \text{else} \end{cases} = \begin{cases} e^{\pi t}e^{-\pi s} & ; s > t-2 \\ 0 & ; \text{else} \end{cases}$$



$f(s)g(t-s)$

$\Rightarrow$ 3 cases to consider

Case 1

$t-2 > 0 \Rightarrow t > 2$

$$f(s) \cdot g(t-s) = 0$$

$$h(t) = \int_{-\infty}^{\infty} 0\,ds = 0$$

Case 2

$t-2 < -3 \Rightarrow t < -1$

$$h(t) = \int_{-3}^{0} e^{\pi t} e^{-\pi s}\,ds = e^{\pi t}\left[\frac{e^{-\pi s}}{-\pi}\right]_{-3}^{0}$$

$$h(t) = e^{\pi t}\left[-\frac{1}{\pi} + \frac{e^{3\pi}}{\pi}\right]$$

$$h(t) = e^{\pi t}\left[\frac{e^{3\pi}-1}{\pi}\right]$$

Case 3

$-3 \le t-2 \le 0 \Rightarrow -1 \le t \le 2$

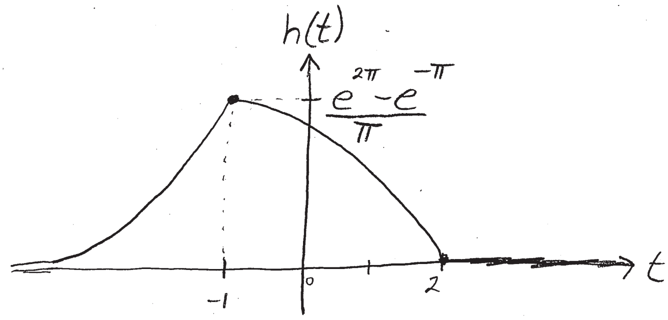
$$h(t) = \int_{t-2}^{0} e^{\pi t} e^{-\pi s}\,ds = e^{\pi t}\left[-\frac{e^{-\pi s}}{\pi}\right]_{t-2}^{0}$$

$$h(t) = e^{\pi t}\left[-\frac{1}{\pi} + \frac{e^{-\pi(t-2)}}{\pi}\right] = \frac{e^{\pi t}e^{-\pi t}e^{2\pi} - e^{\pi t}}{\pi}$$

$$h(t) = \frac{e^{2\pi} - e^{\pi t}}{\pi}$$

$\Rightarrow h(t)$ is a piecewise function

$$h(t)=\begin{cases} e^{\pi t}\left[\dfrac{e^{3\pi}-1}{\pi}\right] & , t<-1 \\ \dfrac{e^{2\pi}-e^{\pi t}}{\pi} & , -1\le t\le 2 \\ 0 & , t>2 \end{cases}$$



Question 26

$$\text{Total Energy} = \int_{-\infty}^{\infty} |X(t)|^2\,dt \qquad \text{Average Power} = \lim_{T\to\infty}\frac{1}{2T}\cdot E$$

- $$E_{f(t)} = \int_{-3}^{0} 1\,dt = t\Big]_{-3}^{0} = 0-(-3) = 3$$

$$\boxed{E_{f(t)} = 3 \quad \text{finite}}$$

$$P_{f(t)} = \lim_{T\to\infty}\frac{1}{2T}(3) = 0 \qquad \boxed{P_{f(t)} = 0 \quad \text{finite}}$$

- $$E_{g(t)} = \int_{-\infty}^{2}\left(e^{\pi t}\right)^2 dt = \frac{1}{2\pi}e^{2\pi t}\Big]_{-\infty}^{2} = \frac{1}{2\pi}\left[e^{4\pi}-e^{-\infty}\right] = \frac{e^{4\pi}}{2\pi}$$

$$\boxed{E_{g(t)} = \frac{e^{4\pi}}{2\pi} \quad \text{finite}}$$

$$P_{g(t)} = \lim_{T\to\infty}\frac{1}{2T}\left(\frac{e^{4\pi}}{2\pi}\right) = 0 \qquad \boxed{P_{g(t)} = 0 \quad \text{finite}}$$

- $$E_{h(t)} = \int_{-\infty}^{-1}\left(e^{\pi t}\left[\tfrac{1}{\pi}(e^{3\pi}-1)\right]\right)^2 dt + \int_{-1}^{2}\left[\tfrac{1}{\pi}(e^{2\pi}-e^{\pi t})\right]^2 dt$$

$$= \left(\tfrac{1}{\pi}(e^{3\pi}-1)\right)^2\left[\frac{e^{2\pi t}}{2\pi}\right]_{-\infty}^{-1} + \frac{1}{\pi^2}\int_{-1}^{2} e^{4\pi}-2e^{2\pi}e^{\pi t}+e^{2\pi t}\,dt$$

$$= \frac{e^{6\pi}-2e^{3\pi}+1}{\pi^2}\cdot\frac{e^{-2\pi}}{2\pi} + \frac{1}{\pi^2}\left[e^{4\pi}t - \frac{2e^{2\pi}}{\pi}e^{\pi t} + \frac{1}{2\pi}e^{2\pi t}\right]_{-1}^{2}$$

$$= \frac{e^{4\pi}-2e^{\pi}+e^{-2\pi}}{2\pi^3} + \frac{1}{\pi^2}\left[\left(2e^{4\pi}-\frac{2e^{2\pi}}{\pi}e^{2\pi}+\frac{1}{2\pi}e^{4\pi}\right)-\left(-e^{4\pi}-\frac{2e^{2\pi}}{\pi}e^{-\pi}+\frac{1}{2\pi}e^{-2\pi}\right)\right]$$

$$= \frac{e^{4\pi} - 2e^{\pi} + e^{-2\pi}}{2\pi^3} + \frac{2e^{4\pi}}{\pi^2} - \frac{2e^{4\pi}}{\pi^3} + \frac{e^{4\pi}}{2\pi^3} + \frac{e^{4\pi}}{\pi^2} + \frac{2e^{\pi}}{\pi^3} - \frac{1}{2\pi^3}e^{-2\pi}$$

$$= \frac{e^{4\pi} - 2e^{\pi} + e^{-2\pi} + 4\pi e^{4\pi} - 4e^{4\pi} + e^{4\pi} + 2\pi e^{4\pi} + 4e^{\pi} - e^{-2\pi}}{2\pi^3}$$

$$= \frac{e^{4\pi}(6\pi - 2) + 2e^{\pi}}{2\pi^3} = \boxed{\frac{e^{4\pi}(3\pi - 1) + e^{\pi}}{\pi^3} \quad \text{finite}}$$

$$P_{h(t)} = \lim_{T\to\infty} \frac{1}{2T}\left(\frac{e^{4\pi}(3\pi-1) + e^{\pi}}{\pi^3}\right) = 0 \qquad \boxed{P_{h(t)} = 0 \quad \text{finite}}$$

Question 27

- $x_1(t) = e^{-|t| - 2j}$

$$E_{x_1(t)} = \int_{-\infty}^{\infty} |e^{-|t|-2j}|^2\, dt = \int_{-\infty}^{\infty} |e^{-|t|} e^{-2j}|^2\, dt = \int_{-\infty}^{\infty} (e^{-|t|})^2\, dt = \int_{-\infty}^{\infty} e^{-2|t|}\, dt$$

$$E_{x_1(t)} = \int_{-\infty}^{0} e^{2t}\, dt + \int_{0}^{\infty} e^{-2t}\, dt$$

$$E_{x_1}(t) = \frac{e^{2t}}{2}\Big]_{-\infty}^{0} + \frac{-e^{-2t}}{2}\Big]_{0}^{\infty} = \left[\left(\frac{1}{2} - 0\right) + \left(0 - \frac{-1}{2}\right)\right] = \frac{1}{2} + \frac{1}{2}$$

$$\boxed{E_{x_1(t)} = 1}$$

$$P_{x_1(t)} = \lim_{T\to\infty} \frac{1}{2T} E_{x_1(t)} = \lim_{T\to\infty} \frac{1}{2T} 1 = 0 \qquad \boxed{P_{x_1(t)} = 0}$$

- $x_2(t) = \sin\left(t + \frac{3\pi}{4}\right)$

$$E_{x_2(t)} = \int_{-\infty}^{\infty} |\sin(t + \tfrac{3\pi}{4})|^2\, dt = \int_{-\infty}^{\infty} \sin^2(t + \tfrac{3\pi}{4})\, dt = \int_{-\infty}^{\infty} \frac{1}{2} - \frac{1}{2}\cos\left(2t + \frac{3\pi}{2}\right) dt$$

note that integrating a cosine/sine over $-\infty$ to ∞ is zero

$$E_{x_2(t)} = \frac{t}{2}\Big]_{-\infty}^{\infty} = \boxed{\infty}$$

$$P_{x_2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\sin\left(t+\frac{3\pi}{4}\right)\right|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{1}{2}\, dt = \lim_{T\to\infty} \frac{1}{2T}\left(\frac{T}{2}+\frac{T}{2}\right) = \lim_{T\to\infty} \frac{1}{2}$$

$$\boxed{P_{x_2(t)} = \frac{1}{2}}$$

- $x_3[n] = e^{-|n-1|}$

$$E_{x_3[n]} = \sum_{n=-\infty}^{\infty} \left|e^{-|n-1|}\right|^2 = \sum_{n=-\infty}^{\infty} e^{-2|n-1|} = \sum_{n=-\infty}^{1} e^{2(n-1)} + \sum_{n=2}^{\infty} e^{-2(n-1)}$$

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let $q = 1-n \Rightarrow n = 1-q$

let $p = n-2 \Rightarrow n = p+2$

$$= \sum_{q=0}^{\infty} e^{2(1-q-1)} + \sum_{p=0}^{\infty} e^{-2(p+2-1)} = \sum_{q=0}^{\infty} e^{-2q} + \sum_{p=0}^{\infty} e^{-2p} e^{-2}$$

$$= \sum_{q=0}^{\infty} \left(\frac{1}{e^2}\right)^q + \frac{1}{e^2} \sum_{p=0}^{\infty} \left(\frac{1}{e^2}\right)^p$$

$$= \frac{1}{1-\frac{1}{e^2}} + \frac{1}{e^2}\left(\frac{1}{1-\frac{1}{e^2}}\right) = \frac{e^2}{e^2-1} + \frac{1}{e^2-1}$$

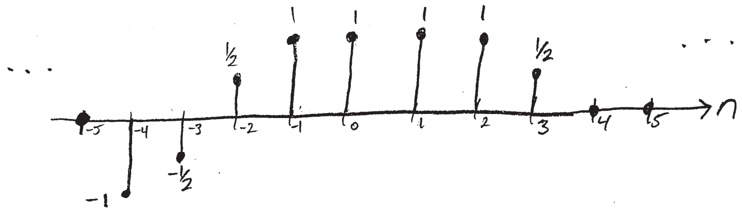
$$\boxed{E_{x_3[n]} = \frac{e^2+1}{e^2-1}}$$

$$P_{x_3[n]} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_3[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1}\left(\frac{e^2+1}{e^2-1}\right) = 0$$

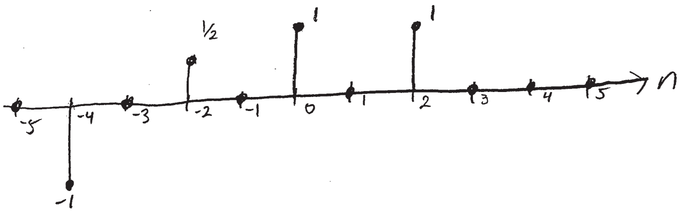
$$\boxed{P_{x_3[n]} = 0}$$

Question 28

1.22)

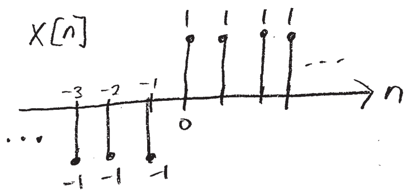


g.) $\frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n] \Rightarrow \begin{cases} x[n] & , n \text{ even} \\ 0 & , n \text{ odd} \end{cases}$



1.24)

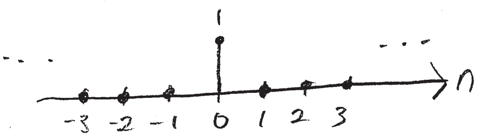
a.) x[n]



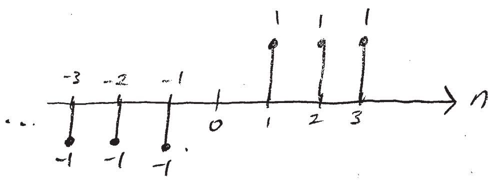
note

$$\begin{cases} X_e[n] = \frac{1}{2}x[n] + \frac{1}{2}x[-n] \\ X_o[n] = \frac{1}{2}x[n] - \frac{1}{2}x[-n] \end{cases}$$

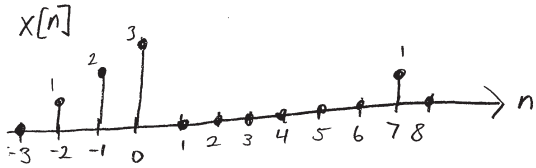
$X_e[n]$



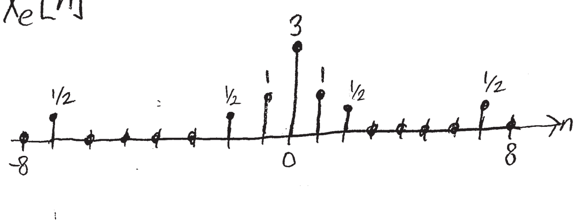
$X_o[n]$



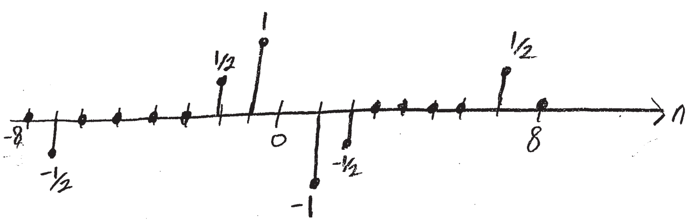
b.) x[n]



$X_e[n]$



$X_o[n]$



Question 29

$\sin\left(t+\frac{\pi}{4}\right)$

$x_e(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$

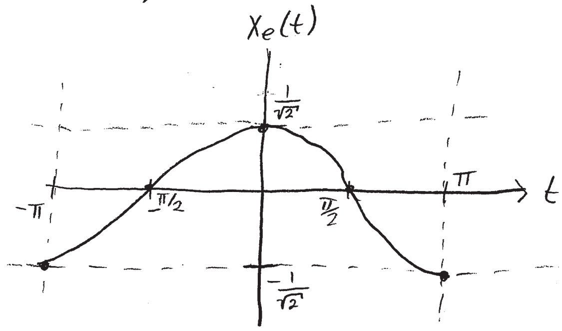
$= \frac{1}{2}\sin\left(t+\frac{\pi}{4}\right) + \frac{1}{2}\sin\left(-t+\frac{\pi}{4}\right)$

note $\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \sin\left(\frac{\left(t+\frac{\pi}{4}\right)+\left(\frac{\pi}{4}-t\right)}{2}\right)\cos\left(\frac{\left(t+\frac{\pi}{4}\right)-\left(\frac{\pi}{4}-t\right)}{2}\right)$

$= \sin\left(\frac{\pi}{4}\right)\cos(t)$

$\boxed{x_e(t) = \frac{1}{\sqrt{2}}\cos(t)}$



Question 30

1.26)

a.) $x[n] = \sin\left(\frac{6\pi}{7}n + 1\right)$

$x[n+N] = \sin\left(\frac{6\pi}{7}(n+N)+1\right) = \sin\left(\frac{6\pi}{7}n + \frac{6\pi N}{7} + 1\right) = x[n]$

$\frac{6\pi N}{7} = 2\pi K \Rightarrow N = \frac{7}{3}K \quad , \quad K \in \mathbb{Z}$

$\boxed{\text{periodic , Fundamental period} = 7}$

c.) $x[n] = \cos(\frac{\pi}{8}n^2)$

$x[n+N] = \cos(\frac{\pi}{8}(n+N)^2) = \cos(\frac{\pi}{8}n^2 + \frac{\pi}{4}nN + \frac{\pi}{8}N^2) \stackrel{?}{=} x[n]$

need $\frac{\pi}{4}nN = 2\pi K_1$ and $\frac{\pi}{8}N^2 = 2\pi K_2$ $\quad K_1, K_2 \in \mathbb{Z}$

$N = \frac{8K_1}{n}$ $\qquad N = 4\sqrt{K_2}$

$\Rightarrow$ if N is a multiple of 8, these expressions hold

$\boxed{\text{periodic, Fundamental period} = 8}$

e.) $x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$

$x[n+N] = 2\cos(\frac{\pi}{4}n + \frac{\pi}{4}N) + \sin(\frac{\pi}{8}n + \frac{\pi}{8}N) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{2}N + \frac{\pi}{6})$

$\frac{\pi}{4}N = 2\pi K_1$ $\qquad \frac{\pi}{8}N = 2\pi K_2$ $\qquad \frac{\pi}{2}N = 2\pi K_3$ $\qquad K_1, K_2, K_3 \in \mathbb{Z}$

$N = 8K_1$ $\qquad N = 16K_2$ $\qquad N = 4K_3$

$LCM(8, 16, 4) = 16$

$\Rightarrow \boxed{\text{periodic, Fundamental Period} = 16}$

Question 30

- $x_k(t) = e^{jk\omega t}, k \in \mathbb{Z} \overset{\omega = \frac{2\pi}{5}}{\Longrightarrow} \boxed{x_k(t) = e^{jk\frac{2\pi}{5}t}}$

There are an $\boxed{\text{infinite number}}$ of distinct CT HRCE's.

Fundamental period $= \frac{2\pi}{|k|\omega} = \frac{2\pi}{|k| \cdot \frac{2\pi}{5}} = \frac{5}{|k|}$

$\Rightarrow \boxed{\text{Common period} = 5}$

- $x_k[n] = e^{jk\omega n}, k \in \mathbb{Z} \overset{\omega = \frac{2\pi}{5}}{\Longrightarrow} \boxed{x_k[n] = e^{jk\frac{2\pi}{5}n}}$

There are $\boxed{N = 5}$ distinct DT HRCE's

Fundamental frequency $= \frac{2\pi}{\text{fundamental period}}$

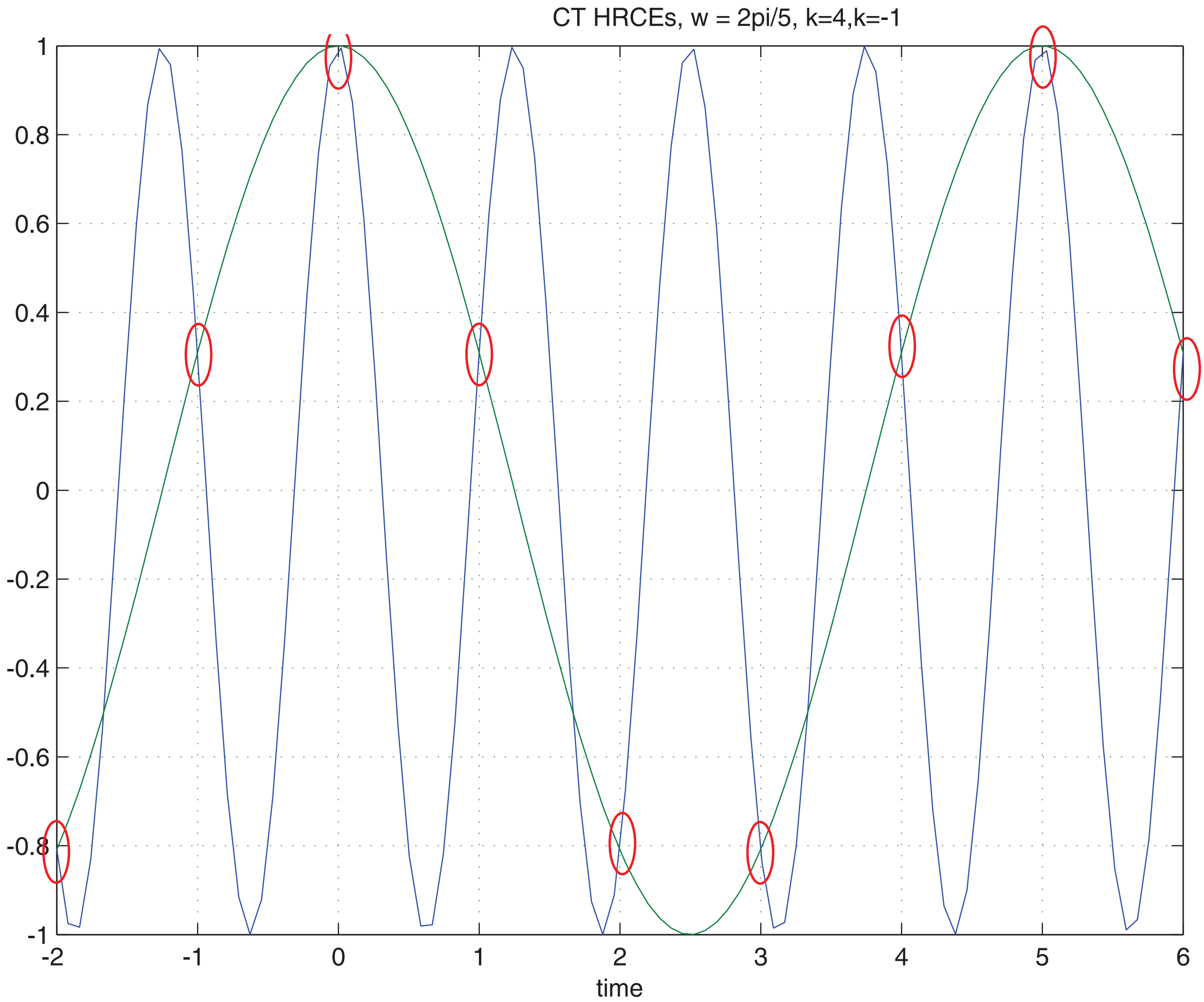
$\frac{2\pi}{5} = \frac{2\pi}{\text{fund. period}} \Rightarrow$ fundamental period $= 5$

$\Rightarrow \boxed{\text{Common period} = 5}$

- Discrete-time signals are continuous time signals sampled at integer values (i.e. n)

If you refer to the matlab plot, you can easily see that the two continuous-time curves intersect whenever t is an integer

$\Rightarrow$ we have lost the ability to distinguish between the curves in discrete time



CT HRCEs, w = 2pi/5, k=4,k=-1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-2
-1
0
1
2
3
4
5
6
time