Question 1

a) $\displaystyle\int_{2\pi}^{3\pi} \sin\left(2w + 0.5\pi + \frac{\pi w}{2}\right) dw = \int_{2\pi}^{3\pi} \sin\left(\frac{4w}{2} + \frac{\pi w}{2} + \frac{\pi}{2}\right) dw$

$$= \int_{2\pi}^{3\pi} \sin\left(\left(\frac{4+\pi}{2}\right)w + \frac{\pi}{2}\right) dw$$

$$= \int_{2\pi}^{3\pi} \cos\left(\left(\frac{4+\pi}{2}\right)w\right) dw$$

$$= \frac{2}{4+\pi} \sin\left(\left(\frac{4+\pi}{2}\right)w\right)\Bigg|_{w=2\pi}^{w=3\pi} = \frac{2}{4+\pi}\left[\sin\left(\frac{12\pi+3\pi^2}{2}\right) - \sin\left(\frac{8\pi+2\pi^2}{2}\right)\right]$$

$$= \boxed{\frac{2}{4+\pi}\left[\sin\left(\frac{3\pi^2}{2}\right) - \sin(\pi^2)\right]} \text{ OR } 0.28\left[\sin(14.8) - \sin(9.87)\right]$$

b) $\displaystyle\int_{0}^{3\pi} \cos\left(\frac{\pi}{3} - u\right) du$

$$= -\sin\left(\frac{\pi}{3} - u\right)\Bigg|_{u=0}^{u=3\pi} = -\left[\sin\left(-\frac{8\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right]$$

$$= \sin\left(\frac{\pi}{3}\right) - \sin\left(-\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) = \boxed{\sqrt{3}}$$

(Q1)

c) $$\int_{y=1}^{2}\int_{x=0}^{2.5} \cos\left(\frac{3x\pi}{4} + \frac{y\pi}{3}\right) dx\,dy$$

$$= \int_{y=1}^{2} \frac{4}{3\pi} \sin\left(\frac{3x\pi}{4} + \frac{y\pi}{3}\right)\Bigg|_{x=0}^{x=2.5} dy$$

$$= \frac{4}{3\pi}\int_{1}^{2} \sin\left(\frac{7.5\pi}{4} + \frac{y\pi}{3}\right) - \sin\left(\frac{y\pi}{3}\right) dy$$

$$= \frac{4}{3\pi}\left[-\frac{3}{\pi}\cos\left(\frac{7.5\pi}{4} + \frac{y\pi}{3}\right) + \frac{3}{\pi}\cos\left(\frac{y\pi}{3}\right)\right]_{y=1}^{y=2}$$

$$= \frac{4}{\pi^2}\left[-\cos\left(\frac{7.5\pi}{4} + \frac{2\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right) - \left[-\cos\left(\frac{7.5\pi}{4} + \frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\right]\right]$$

$$= \frac{4}{\pi^2}\left[-\frac{1}{2} - \frac{1}{2} - \cos\left(\frac{30.5\pi}{12}\right) + \cos\left(\frac{26.5\pi}{12}\right)\right]$$

$$= \boxed{\frac{4}{\pi^2}\left[\cos\left(\frac{53\pi}{24}\right) - \cos\left(\frac{61\pi}{24}\right) - 1\right]}$$

Question 2

a) $$\int_{-2\pi}^{4\pi} 2^{-|t-\pi|} e^{-j\frac{\pi}{2}t} dt = \int_{-2\pi}^{\pi} 2^{t-\pi} e^{-j\frac{\pi}{2}t} dt + \int_{\pi}^{4\pi} 2^{-(t-\pi)} e^{-j\frac{\pi}{2}t} dt$$

$$= 2^{-\pi}\int_{-2\pi}^{\pi} \left(2e^{-j\frac{\pi}{2}}\right)^t dt + 2^{\pi}\int_{\pi}^{4\pi} \left(2^{-1}e^{-j\frac{\pi}{2}}\right)^t dt$$

$$= 2^{-\pi}\left[\frac{\left(2e^{-j\frac{\pi}{2}}\right)^t}{\ln\left(2e^{-j\frac{\pi}{2}}\right)}\right]_{t=-2\pi}^{t=\pi} + 2^{\pi}\left[\frac{\left(\frac{1}{2}e^{-j\frac{\pi}{2}}\right)^t}{\ln\left(\frac{1}{2}e^{-j\frac{\pi}{2}}\right)}\right]_{t=\pi}^{t=4\pi}$$

$$= \frac{1}{2^{\pi}\left(\ln(2) - j\frac{\pi}{2}\right)}\left[2^{\pi} e^{-j\frac{\pi^2}{2}} - 2^{-2\pi} e^{j\pi^2}\right] + \frac{2^{\pi}}{\ln\left(\frac{1}{2}\right) - j\frac{\pi}{2}}\left[2^{-4\pi} e^{-j2\pi^2} - 2^{-\pi} e^{-j\frac{\pi^2}{2}}\right]$$

(Q2, part a)

$$= \frac{e^{-j\frac{\pi^2}{2}}}{\ln(2)-j\frac{\pi}{2}} - \frac{2^{-2\pi}e^{j\pi^2}}{2^{\pi}(\ln(2)-j\frac{\pi}{2})} + \frac{2^{\pi}2^{-4\pi}e^{-j2\pi^2}}{-\ln(2)-j\frac{\pi}{2}} - \frac{e^{-j\frac{\pi^2}{2}}}{-\ln(2)-j\frac{\pi}{2}}$$

$$= \boxed{\frac{e^{-j\frac{\pi^2}{2}} - 2^{-3\pi}e^{j\pi^2}}{\ln(2)-j\frac{\pi}{2}} + \frac{e^{-j\frac{\pi^2}{2}} - 2^{-3\pi}e^{-j2\pi^2}}{\ln(2)+j\frac{\pi}{2}}}$$

b) $\int_{-2}^{3} \omega\cos\left(\frac{\pi}{3}\omega\right)d\omega$ $\quad u = \omega \quad dv = \cos\left(\frac{\pi}{3}\omega\right)d\omega$

$du = d\omega \quad v = \frac{3}{\pi}\sin\left(\frac{\pi}{3}\omega\right)$

$$= uv - \int v\,du$$

$$= \frac{3}{\pi}\left[\omega\sin\left(\frac{\pi}{3}\omega\right)\right]_{\omega=-2}^{\omega=3} - \frac{3}{\pi}\int_{-2}^{3}\sin\left(\frac{\pi}{3}\omega\right)d\omega$$

$$= \frac{3}{\pi}\left[3\,\cancel{\sin(\pi)}^{\,0} - 2\sin\left(\frac{-2\pi}{3}\right)\right] + \frac{9}{\pi^2}\left[\cos\left(\frac{\pi}{3}\omega\right)\right]_{\omega=-2}^{\omega=3}$$

$$= -\left(\frac{6}{\pi}\right)\left(-\frac{\sqrt{3}}{2}\right) + \frac{9}{\pi^2}\left[\overset{-1}{\cos(\pi)} - \cos\left(\frac{2\pi}{3}\right)\right]$$

$$= \frac{3\sqrt{3}}{\pi} - \frac{9}{\pi^2} + \frac{9}{2\pi^2} = \boxed{\frac{6\sqrt{3}\pi - 9}{2\pi^2}}$$

c) $\int_{-3}^{3} (|z-2|)e^{j2\pi z}dz = \int_{-3}^{2} -(z-2)e^{j2\pi z}dz + \int_{2}^{3}(z-2)e^{j2\pi z}dz$

$u = 2-z \quad dv = e^{j2\pi z}dz \qquad p = z-2 \quad dq = e^{j2\pi z}dz$

$du = -dz \quad v = \frac{1}{j2\pi}e^{j2\pi z} \qquad dp = dz \quad q = \frac{1}{j2\pi}e^{j2\pi z}$

$$= uv - \int v\,du + pq - \int q\,dp$$

(Q2, part c)

$$= \frac{2-z}{j2\pi} e^{j2\pi z}\Bigg|_{z=-3}^{z=2} + \frac{1}{j2\pi}\int_{-3}^{2} e^{j2\pi z}\,dz + \frac{z-2}{j2\pi} e^{j2\pi z}\Bigg|_{z=2}^{z=3} - \frac{1}{j2\pi}\int_{2}^{3} e^{j2\pi z}\,dz$$

$$= \left[0 - \frac{5}{j2\pi}e^{-j6\pi}\right] + \frac{1}{\cancel{j^2}4\pi^2} e^{j2\pi z}\Bigg|_{z=-3}^{z=2} + \left[\frac{e^{j6\pi}}{j2\pi} - 0\right] - \frac{1}{\cancel{j^2}4\pi^2} e^{j2\pi z}\Bigg|_{2}^{3}$$

$$= -\frac{5e^{-j6\pi}}{j2\pi} - \frac{e^{j4\pi}}{4\pi^2} + \frac{e^{-j6\pi}}{4\pi^2} + \frac{e^{j6\pi}}{j2\pi} + \frac{e^{j6\pi}}{4\pi^2} - \frac{e^{j4\pi}}{4\pi^2}$$

$$= \frac{-5}{j2\pi}\left(\overset{1}{\cos(-6\pi)} + j\overset{0}{\sin(-6\pi)}\right) - \frac{1}{2\pi^2}\left(\overset{1}{\cos(4\pi)} + j\overset{0}{\sin(4\pi)}\right) + \frac{1}{j2\pi}\left(\overset{1}{\cos(6\pi)} + j\overset{0}{\sin(6\pi)}\right)$$

$$+ \frac{1}{4\pi^2}\left(\overset{1}{\cos(-6\pi)} + j\overset{0}{\sin(-6\pi)}\right) + \frac{1}{4\pi^2}\left(\overset{1}{\cos(6\pi)} + j\overset{0}{\sin(6\pi)}\right)$$

$$= -\frac{5}{j2\pi} - \frac{1}{2\pi^2} + \frac{2}{4\pi^2} + \frac{1}{j2\pi} = \frac{-4}{j2\pi} = \boxed{\frac{-2}{j\pi}} \text{ OR } \frac{2j}{\pi}$$

Question 3

a) $\int_1^4 |\cos(\omega) - j(2\omega+1)|^2 d\omega$

note: $|\cos(\omega) - j(2\omega+1)| = \sqrt{\cos^2(\omega) + (2\omega+1)^2}$

$$= \int_1^4 \cos^2(\omega)\,d\omega + \int_1^4 (2\omega+1)^2\,d\omega$$

$$= \int_1^4 \left(\frac{1}{2} + \frac{1}{2}\cos(2\omega)\right)d\omega + \int_1^4 (4\omega^2 + 4\omega + 1)\,d\omega$$

$$= \frac{1}{2}\omega + \frac{1}{4}\sin(2\omega)\Bigg|_{\omega=1}^{\omega=4} + \left[\frac{4}{3}\omega^3 + 2\omega^2 + \omega\right]_{\omega=1}^{\omega=4}$$

$$= \cancel{2} + \frac{1}{4}\sin(8) - \frac{1}{2} - \frac{1}{4}\sin(2) + \frac{256}{3} + 32 + 4 - \frac{4}{3} - \cancel{2} - 1$$

$$= \boxed{\frac{1}{4}\left(\sin(8) - \sin(2)\right) + \frac{711}{6}}$$

(Q3)

note: $|\sigma + j\omega| = \sqrt{\sigma^2+\omega^2}$

b) $\displaystyle\int_{t=3}^{5}\int_{s=3}^{4} \left|\frac{1}{(2-t)+j2\sqrt{2}t}\right|^2 ds\,dt$

$$= \int_3^5\int_3^4 \frac{1}{(2-t)^2+(2\sqrt{2}t)^2}\,ds\,dt = \int_3^5\int_3^4 \frac{ds\,dt}{t^2-4t+4+8t} = \int_3^5\int_3^4 \frac{ds\,dt}{t^2+4t+4}$$

$$= \int_3^5 \left.\frac{s}{t^2+4t+4}\right|_{s=3}^{s=4} dt = \int_3^5 \frac{1}{t^2+4t+4}\,dt = \int_3^5 \frac{1}{(t+2)^2}\,dt = \int_3^5 (t+2)^{-2}\,dt$$

$$= -(t+2)^{-1}\Big|_{t=3}^{t=5} = \frac{-1}{7} - \frac{-1}{5} = \frac{7}{35} - \frac{5}{35} = \boxed{\frac{2}{35}}$$

Question 4

note: $e^{-j2\pi} = \overset{1}{\cos(-2\pi)} + j\overset{0}{\sin(-2\pi)}$
$= 1$

a) $\displaystyle\int_0^{2/\omega} (t+2s)\, e^{-j\omega\pi(2t+s)}\, ds$

$$= \int_0^{2/\omega} \left(te^{-j\omega\pi(2t+s)} + 2se^{-j\omega\pi(2t+s)}\right) ds$$

$$= te^{-j\omega2\pi t}\int_0^{\frac{2}{\omega}} e^{-j\omega\pi s}\,ds + 2e^{-j\omega2\pi t}\int_0^{\frac{2}{\omega}} se^{-j\omega\pi s}$$

$u = s \quad dv = e^{-j\omega\pi s}$

$du = ds \quad v = \dfrac{-e^{-j\omega\pi s}}{j\omega\pi}$

$$= te^{-j\omega2\pi t}\int_0^{\frac{2}{\omega}} e^{-j\omega\pi s}\,ds + 2e^{-j\omega2\pi t}\left[-\frac{se^{-j\omega\pi s}}{j\omega\pi}\Big|_{s=0}^{s=\frac{2}{\omega}} + \frac{1}{j\omega\pi}\int_0^{2/\omega} e^{-j\omega\pi s}\,ds\right]$$

$$= \left(te^{-j\omega2\pi t} + \frac{2e^{-j\omega2\pi t}}{j\omega\pi}\right)\int_0^{2/\omega} e^{-j\omega\pi s}\,ds + 2e^{-j\omega2\pi t}\left[-\frac{\frac{2}{\omega}\overset{1\text{ (see note)}}{e^{-j2\pi}}}{j\omega\pi}\right]$$

$$= \left(te^{-j\omega2\pi t} + \frac{2e^{-j\omega2\pi t}}{j\omega\pi}\right)\left[\frac{-e^{-j\omega\pi s}}{j\omega\pi}\right]_{s=0}^{s=\frac{2}{\omega}} - \frac{4e^{-j\omega2\pi t}}{j\omega^2\pi}$$

$$= \left(te^{-j\omega2\pi t} + \frac{2e^{-j\omega2\pi t}}{j\omega\pi}\right)\underbrace{\left(-\frac{\overset{1}{e^{-j2\pi}}}{j\omega\pi} - \left(\frac{-1}{j\omega\pi}\right)\right)}_{0} - \frac{4e^{-j\omega2\pi t}}{j\omega^2\pi}$$

$$= -\frac{4e^{-j2\pi\omega t}}{j\omega^2\pi} \quad \text{OR} \quad \boxed{\frac{4je^{-j2\pi\omega t}}{\omega^2\pi}}$$

(Q4)

b) $\frac{1}{T}\left(\int_{-T}^{T/2} (\frac{T}{2} - t)\, e^{j\frac{2\pi k t}{T}} dt + \int_{T/2}^{2T} (t - \frac{T}{2})\, e^{j\frac{2\pi k t}{T}} dt\right)$

$$= \frac{1}{T}\left(\frac{T}{2}\right)\int_{-T}^{T/2} e^{j\frac{2\pi k t}{T}} dt - \frac{1}{T}\int_{-T}^{T/2} t e^{j\frac{2\pi k t}{T}} dt + \frac{1}{T}\int_{\frac{T}{2}}^{2T} t e^{j\frac{2\pi k t}{T}} dt - \frac{1}{T}\left(\frac{T}{2}\right)\int_{\frac{T}{2}}^{2T} e^{j\frac{2\pi k t}{T}} dt$$

$u = \frac{t}{T}$ $\quad dv = e^{j\frac{2\pi k t}{T}}$

$du = \frac{dt}{T}$ $\quad v = \frac{T}{j2\pi k} e^{j\frac{2\pi k t}{T}}$

$$= \frac{1}{2}\int_{-T}^{T/2} e^{j\frac{2\pi k t}{T}} dt - \left[\frac{t}{j2\pi k} e^{j\frac{2\pi k t}{T}}\Big|_{t=-T}^{t=\frac{T}{2}} - \frac{1}{j2\pi k}\int_{-T}^{T/2} e^{j\frac{2\pi k t}{T}} dt\right] + \left[\frac{t}{j2\pi k} e^{j\frac{2\pi k t}{T}}\Big|_{t=\frac{T}{2}}^{t=2T} - \frac{1}{j2\pi k}\int_{\frac{T}{2}}^{2T} e^{j\frac{2\pi k t}{T}} dt\right]$$

$$- \frac{1}{2}\int_{T/2}^{2T} e^{j\frac{2\pi k t}{T}} dt$$

$$= \left(\frac{1}{2} + \frac{1}{j2\pi k}\right)\int_{-T}^{T/2} e^{j\frac{2\pi k t}{T}} dt - \left(\frac{1}{2} + \frac{1}{j2\pi k}\right)\int_{\frac{T}{2}}^{2T} e^{j\frac{2\pi k t}{T}} dt$$

$$- \left[\frac{T}{2 \cdot j2\pi k} e^{j\pi k} + \frac{T}{j2\pi k} e^{j2\pi k}\right] + \left[\frac{T}{j\pi k} e^{j4\pi k} - \frac{T}{2 \cdot j2\pi k} e^{j\pi k}\right]$$

$e^{j\pi k} = \cos(\pi k) + j\sin(\pi k) = (-1)^k$ since k is an integer

$e^{j2\pi k} = \cos(2\pi k) + j\sin(2\pi k) = 1$

$$= \left(\frac{1}{2} + \frac{1}{j2\pi k}\right)\left[\frac{T}{j2\pi k} e^{j\frac{2\pi k t}{T}}\Big|_{t=-T}^{t=\frac{T}{2}} - \frac{T}{j2\pi k} e^{j\frac{2\pi k t}{T}}\Big|_{t=\frac{T}{2}}^{t=2T}\right]$$

$$- \frac{(-1)^k T}{j4\pi k} - \frac{T}{j2\pi k} + \frac{T}{j\pi k} - \frac{(-1)^k T}{j4\pi k}$$

(Q4, part b)

$$= \left(\frac{1}{2}+\frac{1}{j2\pi k}\right)\left[\frac{T}{j2\pi k}e^{j\pi k} - \frac{T}{j2\pi k}e^{-j2\pi k} - \frac{T}{j2\pi k}e^{j4\pi k} + \frac{T}{j2\pi k}e^{j\pi k}\right] - \frac{(-1)^k T}{j4\pi k} - \frac{T}{j2\pi k} + \frac{T}{j\pi k} - \frac{(-1)^k T}{j4\pi k}$$

$$= \frac{T(-1)^k}{j2\pi k} + \frac{2T(-1)^k}{(j2\pi k)^2} - \frac{T}{j2\pi k} - \frac{2T}{(j2\pi k)^2} - \frac{(-1)^k T}{j4\pi k} - \frac{T}{j2\pi k} + \cancel{\frac{T}{j\pi k}} - \frac{(-1)^k T}{j4\pi k}$$

$$= \cancel{\frac{T(-1)^k}{j2\pi k}} - \cancel{\frac{2(-1)^k T}{j4\pi k}} + \frac{2T(-1)^k}{(j2\pi k)^2} - \frac{2T}{(j2\pi k^2)} - \cancel{\frac{2T}{j2\pi k}} + \cancel{\frac{T}{j\pi k}}$$

$$= \boxed{\left[(-1)^k - 1\right]\frac{2T}{(j2\pi k)^2}}$$

Question 5 $\quad f(s) = s^2 + 2s - 3 \quad g(t) = f(1-t)$

$g(t) = (1-t)^2 + 2(1-t) - 3$

$\quad = 1 - 2t + t^2 + 2 - 2t - 3 = t^2 - 4t$

$g'(t) = \frac{d}{dt}\left[t^2 - 4t\right] = 2t - 4$

a) $g(3) = 9 - 4(3)$

$\boxed{g(3) = -3}$

b) $\int_2^5 g(1+2s)\,ds$

$$g(1+2s) = (1+2s)^2 - 4(1+2s) = 1 + 4s + 4s^2 - 4 - 8s$$
$$= 4s^2 - 4s - 3$$

$$\int_2^5 4s^2 - 4s - 3\,ds = \frac{4}{3}s^3 - 2s^2 - 3s\Big|_{s=2}^{s=5}$$

$$= \frac{4}{3}(125) - 2(25) - 3(5) - \frac{4}{3}(8) + 2(4) - 3(2)$$

$$= 156 - 42 - 9 = \boxed{105}$$

c) $\int_0^3 g'(s)e^{j\pi s}\,ds = \int_0^3 2se^{j\pi s} - 4\int_0^3 e^{j\pi s}$

$$u = 2s \qquad dv = e^{j\pi s}\,ds$$
$$du = 2ds \qquad v = \frac{e^{j\pi s}}{j\pi}$$

$$= \frac{2se^{j\pi s}}{j\pi}\Big|_{s=0}^{s=3} - \frac{2}{j\pi}\int_0^3 e^{j\pi s}\,ds - \frac{4}{j\pi}e^{j\pi s}\Big|_{s=0}^{s=3}$$

$$= \frac{6e^{j3\pi}}{j\pi} - 0 - \left[\frac{2}{(j\pi)^2}e^{j\pi s}\right]_{s=0}^{s=3} - \frac{4e^{j3\pi}}{j\pi} + \frac{4\cdot 1}{j\pi}$$

$$e^{j3\pi} = \overset{-1}{\cos(3\pi)} + j\overset{0}{\sin(3\pi)} = -1$$

$$= \frac{-6}{j\pi} - \left[\frac{2(-1)}{j^2\pi^2} - \frac{2\cdot 1}{j^2\pi^2}\right] + \frac{4}{j\pi} + \frac{4}{j\pi}$$

$$= \boxed{\frac{2}{j\pi} - \frac{4}{\pi^2}}$$

Question 6 $\quad f(t) = |1-t| \quad g(t) = \int_{t-1}^{t+2} f(s)\,ds$

a) $g(3) = \int_2^5 |1-s|\,ds = \int_2^5 -(1-s)\,ds = \int_2^5 (s-1)\,ds$

$= \frac{1}{2}s^2 - s \Big|_2^5 = \frac{25}{2} - 5 - 4 + 2 = \boxed{\frac{15}{2}}$

b) $\int_1^4 g(s)\,ds = \int_1^4 \int_{s-1}^{s+2} |1-t|\,dt\,ds$

case 1: $s+2 \le 1 \rightarrow s \le -1$ (don't care)

case 2: $s+2 > 1$ and $s-1 < 1 \longrightarrow -1 \le s < 2$

case 3: $s-1 \ge 1 \longrightarrow s \ge 2$

so $\int_1^4 g(s)\,ds = \int_1^2 \left(\int_{s-1}^{1} (1-t)\,dt + \int_1^{s+2} (t-1)\,dt \right) ds + \int_2^4 \int_{s-1}^{s+2} (t-1)\,dt\,ds$

$= \int_1^2 \left(\left[t - \frac{t^2}{2} \right]_{t=s-1}^{t=1} + \left[\frac{t^2}{2} - t \right]_{t=1}^{t=s+2} \right) ds + \int_2^4 \left[\frac{t^2}{2} - t \right]_{t=s-1}^{t=s+2} ds$

$= \int_1^2 \left(1 - \frac{1}{2} - s + 1 + \frac{(s-1)^2}{2} + \frac{(s+2)^2}{2} - s + 2 - \frac{1}{2} + 1 \right) ds + \int_2^4 \left(\frac{(s+2)^2}{2} - s + 2 - \frac{(s-1)^2}{2} + s - 1 \right) ds$

$= \int_1^2 \left(s^2 - s + \frac{5}{2} \right) ds + \int_2^4 \left(3s - \frac{3}{2} \right) ds$

$= \left[\frac{s^3}{3} - \frac{s^2}{2} + \frac{5s}{2} \right]_{s=1}^{s=2} + \left[\frac{3s^2}{2} - \frac{3s}{2} \right]_{s=2}^{s=4}$

$= \frac{8}{3} - 2 + 5 - \frac{1}{3} + \frac{1}{2} - \frac{5}{2} + 24 - 6 - 6 + 3$

$= \boxed{\frac{55}{3}}$

(Q6)

c) $\int_{-1}^{1} f(s)\,g(1-s)\,ds = \int_{-1}^{1} |1-s| \int_{-s}^{3-s} |1-t|\,dt\,ds$

case 1: $3-s \le 1 \rightarrow s \ge 2$ (don't care)

case 2: $3-s > 1$ and $-s \le 1 \rightarrow -1 \le s < 2$

case 3: $-s > 1 \rightarrow s < -1$ (don't care)

$$= \int_{-1}^{1} |1-s| \left(\int_{-s}^{1} (1-t)\,dt + \int_{1}^{3-s} (t-1)\,dt \right) ds$$

$$= \int_{-1}^{1} |1-s| \left[\left(t - \frac{t^2}{2} \right)\Big|_{t=-s}^{t=1} + \left(\frac{t^2}{2} - t \right)\Big|_{t=1}^{t=3-s} \right] ds$$

$$= \int_{-1}^{1} |1-s| \left(\frac{1}{2} + s + \frac{s^2}{2} + \frac{1}{2}s^2 - 3s + \frac{9}{2} - 3 + s + \frac{1}{2} \right) ds$$

$$= \int_{-1}^{1} |1-s| \left(s^2 - s + \frac{5}{2} \right) ds = \int_{-1}^{1} (1-s)\left(s^2 - s + \frac{5}{2} \right) ds$$

$$= \int_{-1}^{1} -s^3 + 2s^2 - \frac{7}{2}s + \frac{5}{2}\,ds = \frac{-s^4}{4} + \frac{2s^3}{3} - \frac{7}{4}s^2 + \frac{5}{2}s \Big|_{-1}^{1}$$

$$= -\frac{1}{4}(1-1) + \frac{2}{3}(1+1) - \frac{7}{4}(1-1) + \frac{5}{2}(1+1)$$

$$= \frac{4}{3} + 5$$

$$= \boxed{\frac{19}{3}}$$

Question 7 $\quad f[n] = 2n+1 \qquad g[n] = \sum_{k=-2}^{1} k f[k-n]$

$$g[n] = \sum_{k=-2}^{1} k\Big(2(k-n)+1\Big) = \sum_{k=-2}^{1} 2k^2 - 2kn + k$$

a) $g[3] = $

$$\begin{array}{rl} 2(4) - 2(-2)(3) - 2 & (k=-2) \\ 2(1) - 2(-1)(3) - 1 & (k=-1) \\ 0 + 0 + 0 & (k=0) \\ +\ 2(1) - 2(1)(3) + 1 & (k=1) \\ \hline 12 + 12 - 2 & \end{array}$$

$$= \boxed{22}$$

b) $$\sum_{k=-2}^{1} 4^k g[k] = \sum_{k=-2}^{1} 4^k \sum_{\ell=-2}^{1} 2\ell^2 - 2\ell k + \ell$$

$$= \sum_{k=-2}^{1} 4^k (12 + 4k - 2) = \sum_{k=-2}^{1} 4^k (10+4k)$$

$$= 4^{-2}(10+4(-2)) + 4^{-1}(10+4(-1)) + 4^0(10+4(0)) + 4^1(10+4)$$

$$= \frac{2}{16} + \frac{6}{4} + 10 + 14(4)$$

$$= \boxed{\frac{541}{8}}$$

(Q7)

c) $\sum_{k=3}^{\infty} 3^{-k-1} g[k+1]$

substitute: $\ell = k+1$ $(k = \ell - 1)$ $\longrightarrow \sum_{\ell=4}^{\infty} 3^{-\ell} g[\ell]$

$$= \sum_{\ell=4}^{\infty} 3^{-\ell} \sum_{n=-2}^{1} 2n^2 - 2n\ell + n$$

$$= \sum_{\ell=4}^{\infty} \left(\frac{1}{3}\right)^{\ell} (10 + 4\ell) \xrightarrow{\text{back to } k} = \sum_{k=3}^{\infty} \left(\frac{1}{3}\right)^{k+1} (14 + 4k)$$

* need sum to start at 1 to use geometric seq. sum formula

substitute: $m = k - 2$ $(k = m + 2)$

$$= \sum_{m=1}^{\infty} \left(\frac{1}{3}\right)^{m+3} (22 + 4m) = \sum_{m=1}^{\infty} \left(\frac{1}{3}\right)^{m-1} \left(\frac{1}{3}\right)^{4} (22 + 4m)$$

$$= 22\left(\frac{1}{3}\right)^4 \sum_{m=1}^{\infty} \left(\frac{1}{3}\right)^{m-1} + 4\left(\frac{1}{3}\right)^4 \sum_{m=1}^{\infty} m \left(\frac{1}{3}\right)^{m-1}$$

$$= \frac{22}{81}\left(\frac{1}{1-\frac{1}{3}}\right) + \frac{4}{81}\left(\frac{1}{(1-\frac{1}{3})^2}\right) = \frac{22}{81}\left(\frac{3}{2}\right) + \frac{4}{81}\left(\frac{9}{4}\right)$$

$$= \frac{33}{81} + \frac{9}{81} = \boxed{\frac{14}{27}}$$

(Q7)

(from previous parts of question)

d) $\sum_{k=-\infty}^{-2} 2^k g[k] = \sum_{k=-\infty}^{-2} 2^k (10+4k)$

substitute: $m = -k-1$
$(k = -m-1)$

$$= \sum_{m=1}^{\infty} 2^{-m-1}(6-4m) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m+1}(6-4m) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1}\left(\frac{1}{2}\right)^2(6-4m)$$

$$= 6\left(\frac{1}{2}\right)^2 \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} - 4\left(\frac{1}{2}\right)^2 \sum_{m=1}^{\infty} m\left(\frac{1}{2}\right)^{m-1}$$

$$= \frac{\cancel{6}^{3}}{\cancel{4}_{2}}\left(\frac{1}{1-\frac{1}{2}}\right) - \frac{4}{4}\left(\frac{1}{(1-\frac{1}{2})^2}\right) = 3 - 4 = \boxed{-1}$$

Question 8 $\quad f[n] = 3n-1 \quad g[n] = \frac{1}{2}\left(f[n] - f[-n]\right)$

$$g[n] = \frac{1}{2}\left[3n-1-(-3n-1)\right] = \frac{1}{2}(6n) = \underline{3n}$$

a) $g[3] = 3(3) = \boxed{9}$

b) $\sum_{k=-1}^{1} f[k]\,g[2-k] = \sum_{k=-1}^{1} (3k-1)(3(2-k))$

$$= \sum_{k=-1}^{1} (3k-1)(6-3k) = \sum_{k=-1}^{1} -9k^2+21k-6 = 3\sum_{k=-1}^{1} -3k^2+7k-2$$

$$= 3\begin{pmatrix} -3(1) + 7(-1) - 2 \\ 0 + 0 - 2 \\ + -3(1) + 7(1) - 2 \end{pmatrix}$$

$$= 3(-6-6)$$

$$= \boxed{-36}$$

Question 9 $f[n] = \begin{cases} 1 - j|n|, & \text{for } -2 \le n \le 1 \\ 0, & \text{else} \end{cases}$

$$g[n] = \sum_{k=-\infty}^{\infty} f[n-k]\, 2^{|k|}$$

$$f[n-k] = \begin{cases} 1 - j|n-k|, & \text{for } -2 \le n-k \le 1 \rightarrow n-1 \le k \le n+2 \\ 0, & \text{else} \end{cases}$$

$$g[n] = \sum_{k=n-1}^{k=n+2} (1 - j|n-k|)\, 2^{|k|}$$

a) $g[-3] = \sum_{k=-4}^{k=-1} (1 - j|-3-k|)\, 2^{|k|}$

$= (1-j)2^4 + (1-j0)2^3 + (1-j)2^2 + (1-j2)2^1$

$= 16 - 16j + 8 + 4 - 4j + 2 - 4j = \boxed{30 - 24j}$

b) $\boxed{g[n] = \left(2^{|n-1|} + 2^{|n|} + 2^{|n+1|} + 2^{|n+2|}\right) - j\left(2^{|n-1|} + 2^{|n+1|} + 2^{|n+2|+1}\right)}$

c) $h[n] = \sum_{k=-\infty}^{\infty} f[k]\, 2^{|n-k|} = \sum_{k=-2}^{1} (1 - j|k|)\, 2^{|n-k|}$

$h[-3] = (1-2j)2^1 + (1-j)2^2 + (1-j0)2^3 + (1-j)2^4$

$= 2 - 4j + 4 - 4j + 8 + 16 - 16j = \boxed{30 - 24j}$

d) $\boxed{h[n] = \left(2^{|n-1|} + 2^{|n|} + 2^{|n+1|} + 2^{|n+2|}\right) - j\left(2^{|n-1|} + 2^{|n+1|} + 2^{|n+2|+1}\right)}$

(same as part b)

Question 10 $f(t) = \cos(0.5\pi t - \pi)$ $g[n] = f(6n)$

$$g[n] = \cos(0.5\pi(6n) - \pi)$$
$$= \cos(3\pi n - \pi) = -\cos(3\pi n) = \underline{-\cos(\pi n)}$$

$$\boxed{g[0] = -1 \quad g[1] = 1 \quad g[2] = -1 \quad g[3] = 1}$$ $\underline{g[n] = (-1)^{n+1}}$

Question 11 $f[n] = (2n-1)^{-2}$ $g(t) = e^{-j2t}(\cos(t)f[3] + j\sin(t)f[-1])$

$$f[3] = \frac{1}{(2(3)-1)^2} = \frac{1}{25} \quad f[-1] = \frac{1}{(2(-1)-1)^2} = \frac{1}{9}$$

$$\int_{-\pi/2}^{\pi} |g(s)|^2 \, ds$$

$$g(s) = e^{-j2s}\left(\frac{1}{25}\cos(s) + \frac{1}{9}j\sin(s)\right)$$
$$|g(s)| = \sqrt{\cos^2(s)\left(\frac{1}{25}\right)^2 + \sin^2(s)\left(\frac{1}{9}\right)^2}$$

$$= \frac{1}{625}\int_{-\pi/2}^{\pi} \cos^2(s)\,ds + \frac{1}{81}\int_{-\pi/2}^{\pi} \sin^2(s)\,ds$$

$$= \frac{1}{625}\int_{-\pi/2}^{\pi} \frac{1}{2} + \frac{1}{2}\cos(2s)\,ds + \frac{1}{81}\int_{-\pi/2}^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2s)\,ds$$

$$= \frac{1}{625}\left[\frac{s}{2} + \frac{1}{4}\sin(2s)\right]_{s=-\frac{\pi}{2}}^{s=\pi} + \frac{1}{81}\left[\frac{s}{2} - \frac{1}{4}\sin(2s)\right]_{s=-\frac{\pi}{2}}^{s=\pi}$$

$$= \frac{1}{625}\left[\frac{\pi}{2} + 0 - \left(-\frac{\pi}{4} + 0\right)\right] + \frac{1}{81}\left[\frac{\pi}{2} - 0 - \left(-\frac{\pi}{4} - 0\right)\right]$$

$$= \frac{1}{625}\left(\frac{3\pi}{4}\right) + \frac{1}{81}\left(\frac{3\pi}{4}\right) = \frac{3\pi}{4}\left(\frac{706}{50{,}625}\right)$$

$$= \boxed{\frac{3\pi}{2}\left(\frac{353}{50{,}625}\right)}$$

Question 12

a) $\dfrac{1}{5-\omega^2-6j\omega} = \dfrac{1}{j^2\omega^2-6j\omega+5} = \dfrac{1}{(j\omega-5)(j\omega-1)} = \dfrac{a}{j\omega-5} + \dfrac{d}{j\omega-1}$

$a(j\omega-1) + d(j\omega-5) = 1$ $\quad \begin{cases} a+d=0 \\ -a-5d=1 \end{cases}$

$aj\omega - a + dj\omega - 5d = 1$

$-4d = 1 \rightarrow d = -\frac{1}{4} \quad a = \frac{1}{4}$

$\boxed{a=\frac{1}{4} \quad b=-5 \quad c=1 \quad d=-\frac{1}{4} \quad e=-1 \quad f=1}$

b) $\dfrac{3-2j\omega}{(j\omega-5)(j\omega-1)(1-j\omega)} = \dfrac{2j\omega-3}{(j\omega-5)(j\omega-1)^2} = \dfrac{d}{j\omega-5} + \dfrac{a}{j\omega-1} + \dfrac{g}{(j\omega-1)^2}$

$d(j\omega-1)^2 + a(j\omega-5)(j\omega-1) + g(j\omega-5)$

$= d(-\omega^2-2j\omega+1) + a(-\omega^2-6j\omega+5) + g(j\omega-5) = 2j\omega-3$

$\begin{cases} -d-a=0 \\ -2d-6a+g=2 \\ d+5a-5g=-3 \end{cases}$

$a=-d$

$2a-6a+g=2 \rightarrow g=2+4a$

$-a+5a-10-20a=-3$

$-16a=7 \rightarrow a=-\frac{7}{16}$

$g=2+4\left(-\frac{7}{16}\right)=\frac{1}{4}$

$\boxed{\begin{array}{l} a=-\frac{7}{16} \quad b=-1 \quad c=1 \\ d=\frac{7}{16} \quad e=-5 \quad f=1 \\ g=\frac{1}{4} \end{array}}$