ECE 302, Supplemental PDF #2: A special coin-flipping game

1. The minimal bet is \$1M.
2. Flip a coin 1M times. Count the number of heads and then calculate the relative frequency by

$$\text{freq} = \frac{\text{number of heads}}{1,000,000} \tag{1}$$

3. If the computed frequency is between 0.499 and 0.501, then you win \$2M back. Otherwise, you lose the initial bet \$1M.

Question: Should you participate in this game or not?
Answer: Before giving any meaningful answer, we need to decide the weight assignment we use to solve this problem.

Solution 1: Assume that we have a perfectly fair coin with each side having probability $\frac{1}{2}$. In the later part of the semester, we will use a technique called *Gaussian approximation* to compute the probability of winning and the final answer is

$$P(\text{winning}) = P(0.499 < \text{freq} < 0.501) \approx 95.45\%. \tag{2}$$

Yes, we should definitely participate in this game.

Solution 2: Assume that we have a slightly bent coin with (head,tail) probability being $(0.49, 0.51)$. In the later part of the semester, we will use a technique called *Chernoff bound* to compute the probability of winning and the final answer is

$$P(\text{winning}) = P(0.499 < \text{freq} < 0.501) < 4.26 \times 10^{-71}. \tag{3}$$

No, we should NOT participate in this game.

Solution 3: Assume that we have **a near-perfect coin** with (head,tail) probability being $(0.495, 0.505)$. Again, we will use the technique called *Chernoff bound* to compute the probability of winning and the final answer is

$$P(\text{winning}) = P(0.499 < \text{freq} < 0.501) < 1.26 \times 10^{-14}. \tag{4}$$

No, we should NOT participate in this game.

If you do not know the techniques of *Gaussian approximation* and *Chernoff bound*, how are we going to compute the winning probability

$$P(\text{winning}) = P(0.499 < \text{freq} < 0.501)? \tag{5}$$

To explain the principle of computation, we consider a simplified game as follows:

1. The minimal bet is \$5.
2. Flip a coin 5 times. Count the number of heads and then calculate the relative frequency by
$$\text{freq} = \frac{\text{number of heads}}{5} \tag{6}$$
3. If the computed frequency is between 0.3 and 0.7, then you win \$10 back. Otherwise, you lose the initial bet \$5.

Question: what is the probability of winning?

Solution: Assume that we have a perfectly fair coin with each side having probability $\frac{1}{2}$. When flipping a coin 5 times, there are $2^5 = 32$ possibilities and each has weight $\frac{1}{2^5} = \frac{1}{32}$. They can be listed in the following table.

Outcome	Assigned Probability (Weight)	Outcome	Assigned Probability (Weight)
TTTTT	$\frac{1}{32}$	TTTTH	$\frac{1}{32}$
TTTHT	$\frac{1}{32}$	TTTHH	$\frac{1}{32}$
TTHTT	$\frac{1}{32}$	TTHTH	$\frac{1}{32}$
TTHHT	$\frac{1}{32}$	TTHHH	$\frac{1}{32}$
THTTT	$\frac{1}{32}$	THTTH	$\frac{1}{32}$
THTHT	$\frac{1}{32}$	THTHH	$\frac{1}{32}$
THHTT	$\frac{1}{32}$	THHTH	$\frac{1}{32}$
THHHT	$\frac{1}{32}$	THHHH	$\frac{1}{32}$
HTTTT	$\frac{1}{32}$	HTTTH	$\frac{1}{32}$
HTTHT	$\frac{1}{32}$	HTTHH	$\frac{1}{32}$
HTHTT	$\frac{1}{32}$	HTHTH	$\frac{1}{32}$
HTHHT	$\frac{1}{32}$	HTHHH	$\frac{1}{32}$
HHTTT	$\frac{1}{32}$	HHTTH	$\frac{1}{32}$
HHTHT	$\frac{1}{32}$	HHTHH	$\frac{1}{32}$
HHHTT	$\frac{1}{32}$	HHHTH	$\frac{1}{32}$
HHHHT	$\frac{1}{32}$	HHHHH	$\frac{1}{32}$

The probability of winning is thus

$$P(\text{winning}) = P(0.3 < \text{freq} < 0.7) = \frac{1}{32} \cdot 20 = \frac{5}{8}. \tag{7}$$

Task: Please circle the outcomes that are "counted" in our computation.

Obviously the above computation is very time consuming. That is why we need the tools of Gaussian approximation and Chernoff bound when dealing with the first example.