Question 1: [20%, Work-out question, Learning Objective 1] X is a uniform random variable over interval $(0,1)$. Consider the following function

$$f(x) = \begin{cases} \ln(x) & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and we generate a new random variable Y by $Y = f(X)$.

1. [15%] Find $F_Y(y)$, the cdf of Y, and plot $F_Y(y)$ for the range of $-3 \le y \le 3$.
2. [5%] Find $f_Y(y)$, the pdf of Y, and plot $f_Y(y)$ for the range of $-3 \le y \le 3$.

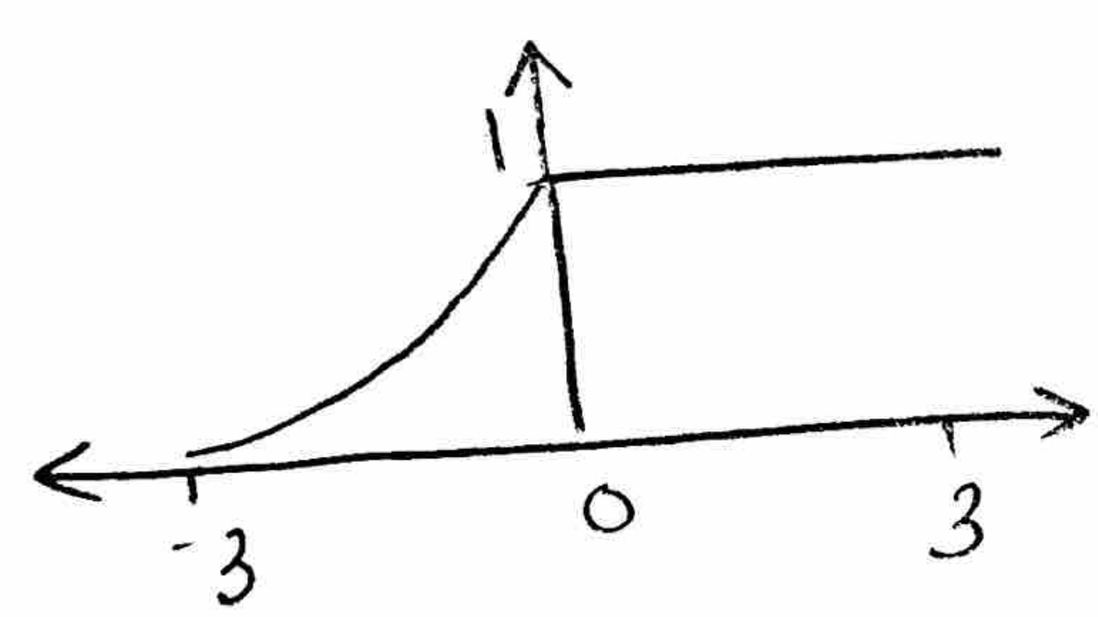
   Hint: If you do not know the answer of the previous sub-question, you can assume that

$$F_Y(y) = \begin{cases} 0 & \text{if } y < -2 \\ \frac{y+3}{6} & \text{if } -2 \le y < 2\,. \\ 1 & \text{if } 2 \le y \end{cases} \tag{2}$$

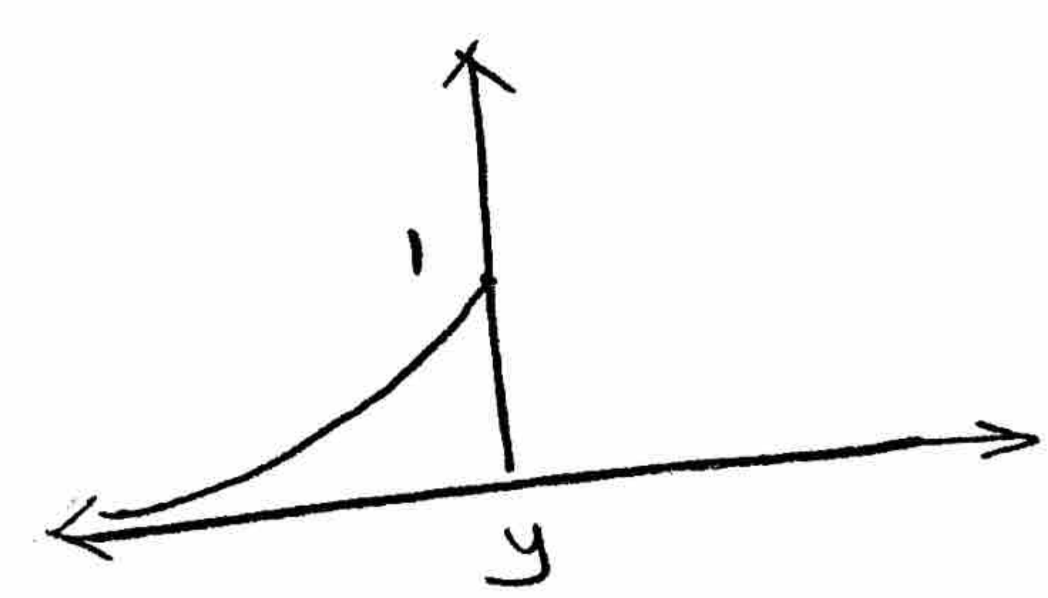
   You will still receive full credit for this sub-question if your answer is right.

① $F_Y(y) = P(Y \le y) = P(f(X) \le y) = P(\ln(X) \le y) = P(X \le e^y)$

$$= \begin{Bmatrix} e^y & -3 \le y \le 0 \\ 1 & 0 \le y \le 3 \end{Bmatrix}$$



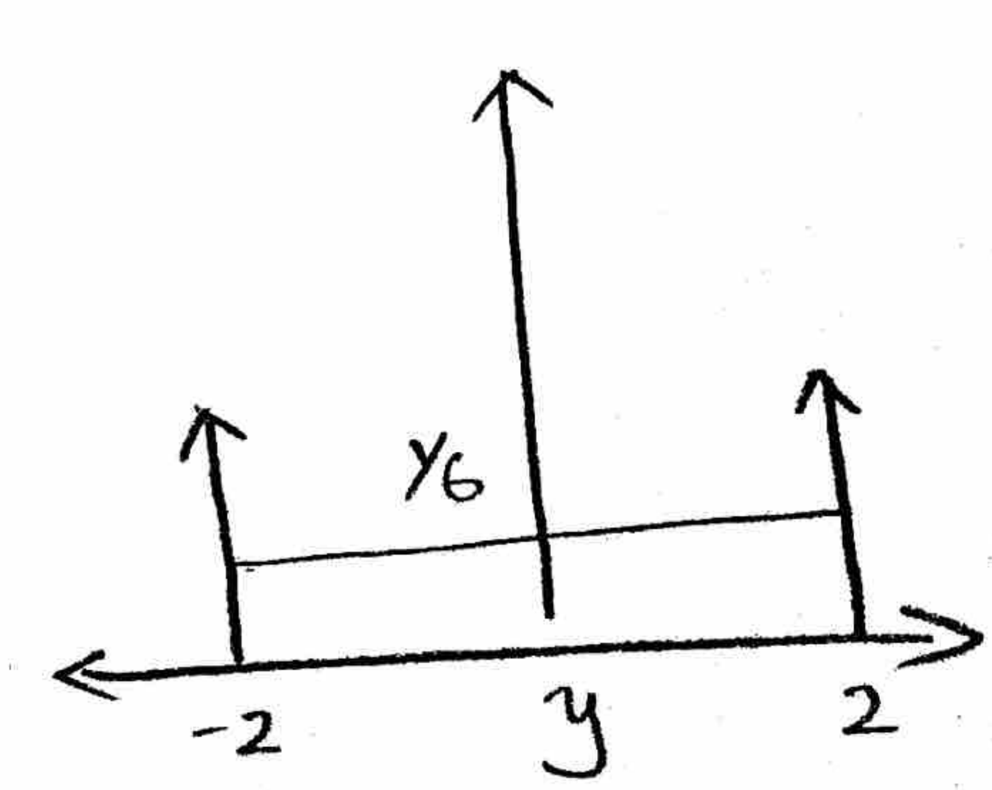
2) $f_Y(y) = \dfrac{\delta F_Y(y)}{\delta y} = \begin{Bmatrix} e^y & y \le 0 \\ 0 & y \geq 0 \end{Bmatrix}$



Alternate solution

$$f_Y(y) = \frac{\delta F_Y(y)}{\delta y} = \begin{Bmatrix} 0 & y < -2 \\ \frac{1}{6} & -2 \le y \le 2 \\ 0 & y > 2 \end{Bmatrix}$$

$$= \frac{1}{6}\delta(y+2) + \frac{1}{6}[u(y+2) - u(y-2)] + \frac{1}{6}\delta(y-2)$$



Question 2: [30%, Work-out question, Learning Objective 1] Consider a exponential random variable X with parameter $\lambda = 3$.

1. [7%] Use the Chebyshev's inequality to bound the probability $P(X \geq 20)$. Hint: If you do not know how to solve this problem, carefully write down what is the Chebyshev inequality. If your answer is correct, you will receive 3.5 points.

2. [7%] Recall that the moment generating function $X^*(s)$ is defined as $X^*(s) = E(e^{-sX})$. Find the expression of $X^*(s)$.

3. [8%] Use the moment theorem to find the *third moment* of X.

   Hint: If you do not know the answer of the previous sub-question, you can assume that $X^*(s) = \frac{1}{1+4s}$. You will still receive full credit for this sub-question if your answer is correct.

4. [8%] Use the Chernoff bound to bound the probability $P(X \geq 20)$.

   Hint 1: If you do not know the expression of $X^*(s)$, you can assume that $X^*(s) = \frac{1}{1+4s}$. You will still receive full credit for this sub-question if your answer is correct.

   Hint 2: Chernoff bound is not easy to use. If you do not know how to answer this question in details, you should first write down what is Chernoff bound and how you "plan" to use the Chernoff bound equation, including how to find the right s value. If your high-level answer is correct, you will receive 4 points even if you do not actually apply the bound with real numbers.

   Hint 3: To receive 8 points, your answer would be of the form like $P(X \geq 20) \leq 0.25e^{-5}$.

① Chebyshev's Inequality $P(|X-m| \geq a) \leq \frac{\sigma^2}{a^2}$

$$P(X \geq 20) = P\left(\left|X-\tfrac{1}{3}\right| \geq 20-\tfrac{1}{3}\right)$$

$$= P\left(\left|X-\tfrac{1}{3}\right| \geq \tfrac{59}{3}\right) \leq \frac{\frac{1}{9}}{\left(\frac{59}{3}\right)^2} = \frac{1}{59^2}$$

2) $X^*(s) = E(e^{-sx})$

$$= \int_0^\infty e^{-sx} \lambda e^{-\lambda x} \, dx$$

$$= \int_0^\infty 3e^{-(3+s)x} \, dx = -\frac{3}{(3+s)} e^{-(3+s)x} \Big|_0^\infty$$

$$= \frac{3}{3+s}$$

3) $E(X^n) = (-1)^n \left[\frac{\delta^n}{\delta s^n} X^*(s) \right]_{s=0}$

$$E(X^3) = (-1)^3 \left[\frac{\delta^3}{\delta s^3} \frac{3}{(s+3)} \right]_{s=0}$$

$$= -1 \times \left. \frac{\delta}{\delta s^2} (-3)(s+3)^{-2} \right]_{s=0}$$

$$= -1 \times \left. \frac{\delta}{\delta s} \; 6(s+3)^{-3} \right]_{s=0}$$

$$= -1 \times \left. \frac{-18}{(s+3)^4} \right]_{s=0} = \frac{+2}{9}$$

4) $P(X \geq a) \leq e^{s \cdot a} X^*(s) = e^{20s} \frac{3}{s+3}$

To find minimum

$$\frac{\delta}{\delta s} \frac{3e^{20s}}{s+3} = \frac{3 \times 20\, e^{20s}}{s+3} + 3e^{20s} (-1)(s+3)^{-2}$$

$$= \frac{3e^{20s}}{s+3} \left[20 - \frac{1}{s+3} \right] \quad — ①$$

Setting eq 1 to zero

$$20 - \frac{1}{s+3} = 0$$

$$20(s+3) = 1$$

$$20s + 60 = 1$$

$$s = -59/20$$

$$\therefore P(x \geq a) \leq e^{-59/20 \times 20} \frac{3}{\frac{-59}{20} + 3}$$

$$= e^{-59}\, 60$$

Question 3: [15%, Work-out question, Learning Objective 1] Consider a continuous Gaussian random variable X with mean 3 and variance 16. A function $f(x)$ takes x as input and outputs the integer part of x. For example, $f(0.333) = 0$, $f(1.975) = 1$, $f(5.002) = 5$, $f(-3.22) = -3$, etc.

Find the conditional probability

$$P(f(X) \text{ is a prime number} | 0 < X < 6) \tag{3}$$

Hint 1: Neither 0 nor 1 is a prime number. The smallest prime number is 2.
Hint 2: The following values may be useful.

$$Q(0) = 0.5 \tag{4}$$
$$Q(0.25) = 0.4013 \tag{5}$$
$$Q(0.5) = 0.3085 \tag{6}$$
$$Q(0.75) = 0.2266 \tag{7}$$

Hint 3: Your answer can be something like $\frac{1-2*0.729+0.883}{0.958+3*0.729+0.883}$. There is no need to further simplify it.

$P(f(X)$ is a prime number $| \ 0 < X < 6)$

$$= \frac{P(f(X) \text{ is a prime} \cap 0 < X < 6)}{P(0<X<6)}$$

Let Z be a normal gaussian distribution

$X = m + \sigma Z = 3 + 4Z$

$$P(0<X<6) = P(0<3+4Z<6) = P\left(-\frac{3}{4}<Z<\frac{3}{4}\right)$$
$$= Q\left(-\frac{3}{4}\right) - Q\left(\frac{3}{4}\right)$$
$$= 1 - Q\left(\frac{3}{4}\right) - Q\left(\frac{3}{4}\right)$$
$$= 1 - 2Q\left(\frac{3}{4}\right)$$
$$= 1 - 2 \times 0.2266$$
$$= 0.5468$$

$P(f(x)$ is a prime number $\&\ 0<x<6)$

$= P(2 \leq X \leq 4) + P(5 \leq X < 6)$

$= P(2 \leq 3+4Z \leq 4) + P(5 \leq 3+4Z < 6)$

$= P(-\frac{1}{4} \leq Z \leq \frac{1}{4}) + P(\frac{1}{2} \leq Z \leq \frac{3}{4})$

$= 1 - Q(+\frac{1}{4}) - Q(\frac{1}{4}) + Q(\frac{1}{2}) - Q(\frac{3}{4})$

$= 1 - 2 \times 0.4013 + 0.3085 - 0.2266$

$$P(f(x) \text{ is a prime number} \mid 0 < X < 6) = \frac{1 - 2 \times 0.4013 + 0.3085 - 0.2266}{0.5468}$$

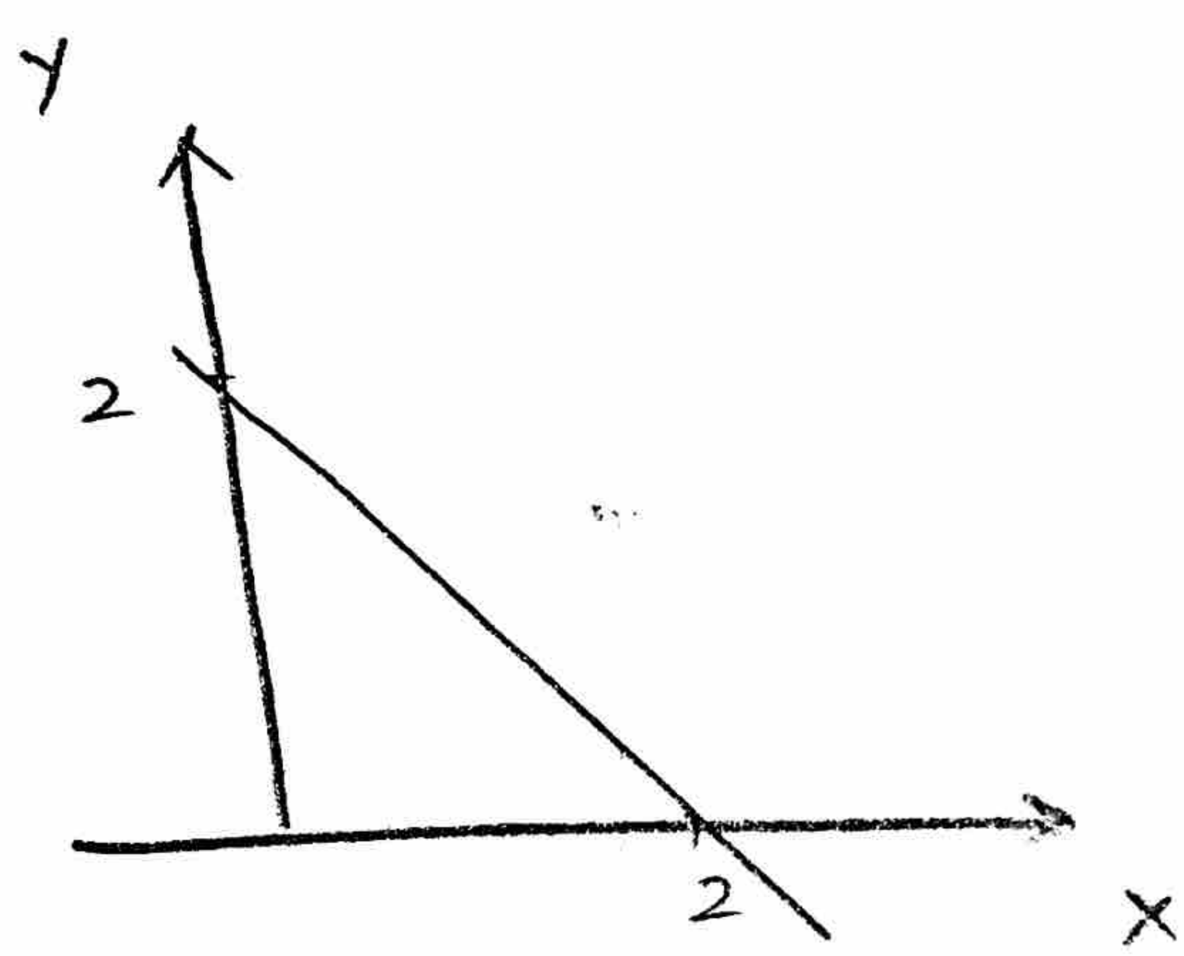
Question 4: [15%, Work-out question, Learning Objective 1] Consider two random variables X and Y with joint pdf

$$f_{XY}(x,y) = \begin{cases} c & \text{if } 0 < x \text{ and } 0 < y \text{ and } x + y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

1. [2%] Find the value of c.

2. [13%] Let $F_{XY}(x,y)$ denote the joint cdf of X and Y. Find the expression of $F_{XY}(x,y)$ assuming both $x < 2$ and $y < 2$. That is, there are usually many cases to consider when deriving the joint cdf. However, in this question you are only asked to solve the cdf for the cases satisfying $x < 2$ and $y < 2$.

①

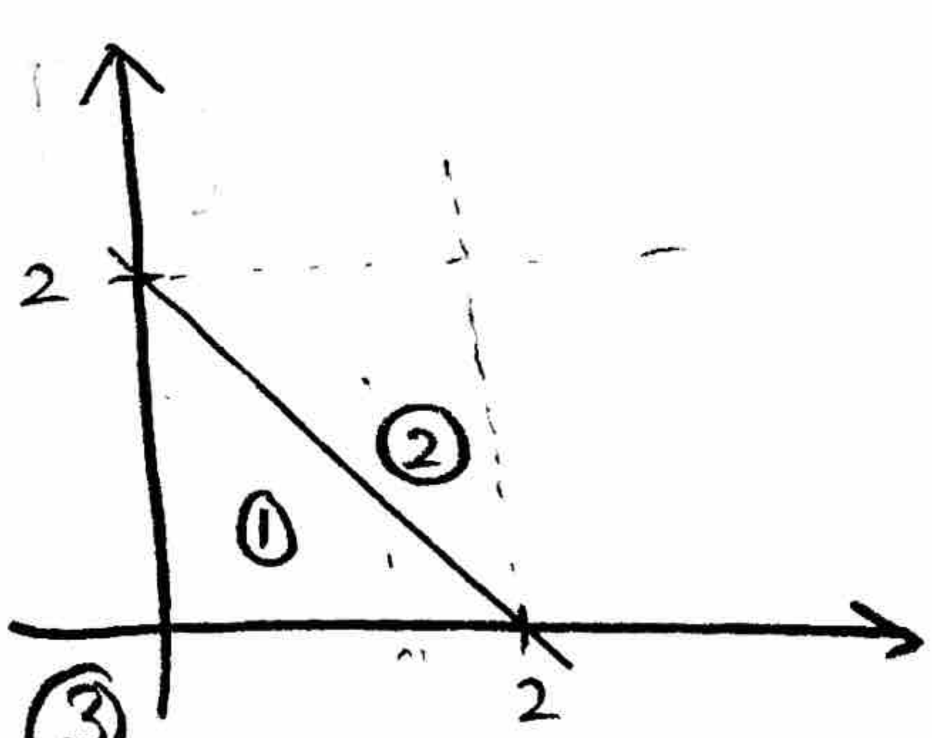


$c \times$ Area of triangle $= 1$

$c \times \frac{1}{2} \times 2 \times 2 = 1$

$c = \frac{1}{2}$

2)

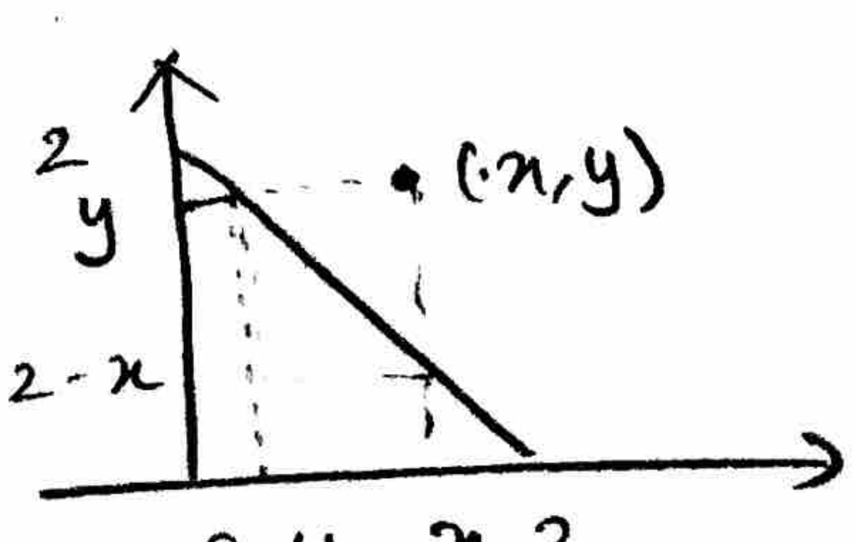


Region 1

$$F_{XY}(x,y) = \int_0^y \int_0^x \frac{1}{2}\, ds\, dt$$

$$= \frac{1}{2} xy$$

Region 2.



$$F_{X,Y}(x,y) = \int_0^{2-x} \int_0^{x} \frac{1}{2}\, ds\, dt + \int_{2-x}^{y} \int_0^{2-t} \frac{1}{2}\, ds\, dt$$

$$= \frac{1}{2} x(2-x) + \int_{2-x}^{y} \frac{1}{2}(2-t)\, dt$$

$$= \frac{1}{2}x(2-x) + \frac{1}{2}\left[2t - \frac{t^2}{2}\right]_{2-x}^{y}$$

$$= x - \frac{x^2}{2} + y - \frac{y^2}{4} - (2-x) + \frac{(2-x)^2}{4}$$

$$= x - \frac{x^2}{2} + y - \frac{y^2}{4} - 2 + x + \frac{4}{4} - x + \frac{x^2}{4}$$

$$= x + y - \frac{x^2}{4} - \frac{y^2}{4} - 1 \,//$$

Region 3.

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} 0 \, ds\,dt$$

$$= 0$$

$$F_{xy}(x,y) = \begin{cases} 0 & x<0 \text{ or } y<0 \\ \frac{xy}{2} & 0<x<2, 0<y<2, x+y<2 \\ x+y-\frac{x^2}{4}-\frac{y^2}{4}-1 & 0<x<2, 0<y<2, x+y>2 \end{cases}$$

Question 5: [20%, Multiple choice question. There is no need to justify your answers]

1. [2%] (X, Y) is uniformly distributed over a rectangle $\{(x, y) : 0 < x < 2, 1 < y < 4\}$. Are X and Y independent?

2. [2%] (X, Y) is uniformly distributed over a unit circle $\{(x, y) : x^2 + y^2 < 1\}$. Are X and Y independent?

3. [2%] Z is Bernoulli distributed with parameter $p = 0.25$, Y is uniformly distributed over range $(-1, 1)$, and Y and Z are independent. Construct the third random variable $X = Y \cdot (2Z - 1)$. Are X and Z independent?

4. [3%] Z is Bernoulli distributed with parameter $p = 0.25$, Y is uniformly distributed over range $(-1, 1)$, and Y and Z are independent. Construct the third random variable $X = Y \cdot (2Z - 1)$. Are X and Y independent?

5. [2%] Consider a random variable X with mean $m = 4$ and variance 50. Is the following statement always true? "By the Markov inequality, we have $P(X \geq 40) \leq \frac{m}{40} = 0.1$."

6. [3%] Suppose X has mean $m_X = 40$ and variance $\sigma_X^2 = 50$. Construct a new random variable $Y = \frac{X}{10}$. Is the following statement always true? "We have $m_Y = 4$ and $\sigma_Y^2 = 5$."

7. [2%] For two random variables X and Y, consider their joint pdf $f_{XY}(x, y)$ and marginal pdf $f_X(x)$ and $f_Y(y)$. Is the following statement always true? "We have $f_{XY}(x, y) \leq f_X(x)$ for all possible x and y."

8. [2%] For two random variables X and Y, consider their joint cdf $F_{XY}(x, y)$ and marginal cdf $F_X(x)$ and $F_Y(y)$. Is the following statement always true? "We have $F_{XY}(x, y) \leq F_X(x)$ for all possible x and y."

9. [2%] X is Poisson distributed with parameter $\alpha_X = 10$. Given $X = x_0$, Y is binomial distributed with $n = x_0$ and $p = 0.5$. Is the following statement always true? "The marginal distribution of Y is Poisson with parameter $\alpha_Y = \alpha_X \cdot 0.5 = 5$."

1. True
2. False
3. True
4. False
5. False
6. False
7 False
8 True
9) True