Midterm #2 of ECE302, Section 2

8–9pm, Monday, February 25, 2019, FRNY G140.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam may contain some multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [10%, Work-out question, Learning Objective 1] X is a binomial random variable with $(n, p) = (5, 0.5)$. Let A denote the event that X is an odd number; and let B denote the event that $X^2 < 5$.

Are events A and B independent or not?

Note: This is not a multiple-choice question. Please carefully write down the detailed reasons why they are independent or not. Just answering whether they are independent or not without any justification will not earn any point for this question.

Question 2: [23%, Work-out question, Learning Objective 1] Consider a geometric random variable X with parameter $p = \frac{1}{3}$. Answer the following questions:

1. [12%] Answer the following three questions:

    (a) Find the probability that "X is a multiple of 2" (i.e., X is even).

    (b) Find the probability that "X is a multiple of 3".

    (c) Find the probability that "X is a multiple of 6".

    Hint 1: 0 is a multiple of 2; 0 is a multiple of 3; and 0 is a multiple of 6.

    Hint 2: It may be easier to find the probability that X is a multiple of m in general and then choose $m = 2, 3,$ and 6, respectively.

2. [4%] Find the probability that "X is neither a multiple of 2 nor a multiple of 3"? You can assume that the answers to the previous 3 sub-questions are b_2, b_3, and b_6, respectively, and express your answer for this question directly using b_2, b_3, and b_6. There is no need to further simplify the answer. You can get full credit even if you do not know the values of b_2, b_3, and b_6.

    Hint 3: You may like to draw the Venn diagram to help your computation.

3. [7%] Find the probability $\mathsf{P}(X > 1.1 | X^2 < 20)$.

    Hint 4: You may leave your answer to be something like $\frac{0.9^{1.5}}{0.2^5+0.4^2+0.3^7}$. There is no need to further simplify it.

Question 3: [20%, Work-out question, Learning Objective 1] Consider a continuous random variable with pdf

$$f(x) = \begin{cases} 0.25x & \text{if } 0 \leq x < 2 \\ 0.5 & \text{if } 2 \leq x < 3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

1. [6%] Find the value of $E(X)$.
2. [5%] Find the value of $\mathsf{Var}(X)$.

   Hint 1: You may leave your answer to be something like $\frac{37}{78} - \frac{15^2}{23}$. There is no need to further simplify it.
3. [5%] Find the value of the third moment of X.

   Hint 2: You may leave your answer to be something like $\frac{37}{78} - \frac{15^2}{23}$. There is no need to further simplify it.
4. [4%] Find the expression of $E(e^{1.25X})$.

   Hint 3: For the last subquestion, you can express $E(e^{1.25X})$ as an integral. There is no need to finish integration. As long as your integral is set up properly, you will receive full credit for the last subquestion. However, for the first 3 questions, you need to finish integration and find the exact values.

Question 4: [20%, Work-out question, Learning Objective 1]

1. [6%] What do the acronyms "pmf", "pdf", and "cdf" stand for?

We consider a random variable X with the following cdf $F(x)$:

$$F(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{x}{12} + \frac{1}{6} & \text{if } 2 < x < 8 \\ 1 & \text{if } 8 \leq x \end{cases} \tag{2}$$

2. [4%] Plot $F(x)$ for the range of $0 \leq x \leq 10$.

3. [2%] Prof. Wang just realized that he forgot to specify the value $F(2)$. It turns out that we must have $F(2) = 1/3$ in this example. Explain why we must have $F(2) = 1/3$. One or two sentences should suffice.

4. [8%] Find the probability that $\mathsf{P}((X-4)^2 \leq 4)$.

Question 5: [27%, Work-out question, Learning Objectives 1 and 2] The next easily observable meteor shower in 2019 will be Eta Aquariid in May 6, 2019. The projected hourly rate is 40 meteors per hour.

Suppose the number of meteors is Poisson distributed. Answer the following questions.

1. [10%] Suppose that I set up a camcorder and record a video clip of 5 minutes. What is the probability that the video clip has recorded at least 6 meteors.

   Hint 1: Your answer can be something like $\frac{3.2^5}{4!} + \frac{2.7^3}{2!}$. There is no need to simplify it.

2. [10%] Suppose that I would like to set up the recording duration to be T unit: minutes. How large my T value needs to be so that with probability 95% my video can record/capture at least 20 meteors.

   Hint 2: A carefully outlined procedure how you plan to find the value of T would suffice.

3. [7%] For any value $y \geq 0$, find out the probability that my waiting time for the first meteor to arrive is $\leq y$ minute. For example, if $y = 2.231$ minutes, then the goal is to find the probability that the first meteor arrives within $y = 2.231$ minutes.

   Since we do not specify the value of y, your answer should be a function of y.

   Hint 3: If you do not know how to solve this problem, you can solve the following question instead:
   Find the probability that the first meteor does not arrive within 20 minutes. You will receive 5 points if your answer is correct.

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\mathrm{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\mathrm{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega} - 1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \le x \le b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.

- 2-dimensional Gaussian Random Vector

  $S = \{(x, y) : \text{for all real-valued } x \text{ and } y\}$

  $$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)}$$

  $E(X) = m_X$, $\mathrm{Var}(X) = \sigma_X^2$, $E(Y) = m_Y$, $\mathrm{Var}(Y) = \sigma_Y^2$, and $\mathrm{Cov}(X, Y) = \rho\sqrt{\sigma_X^2\sigma_Y^2}$.

- n-dimensional Gaussian Random Variable

  $S = \{(x_1, x_2, \cdots, x_n) : \text{for all real-valued } x_1 \text{ to } x_n\}$

  If we denote $\vec{x} = (x_1, x_2, \cdots, x_n)$ as an n-dimensional row-vector, then the pdf of an n-dimensional Gaussian random vector becomes

  $$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(K)}} e^{-\frac{1}{2}(\vec{x}-\vec{m})K^{-1}(\vec{x}-\vec{m})^{\mathrm{T}}}$$

  where $\vec{m}$ is the mean vector of X, i.e., $\vec{m} = E(\vec{X})$; K is an $n \times n$ covariance matrix, where the (i, j)-th entry of the K matrix is $\mathrm{Cov}(X_i, X_j)$; $\det(K)$ is the determinant of K; and K^{-1} is the inverse of K.