Question 1: [12%, Work-out question, Learning Objective 1] Consider a 2-dimensional continuous function $f(x, y)$.

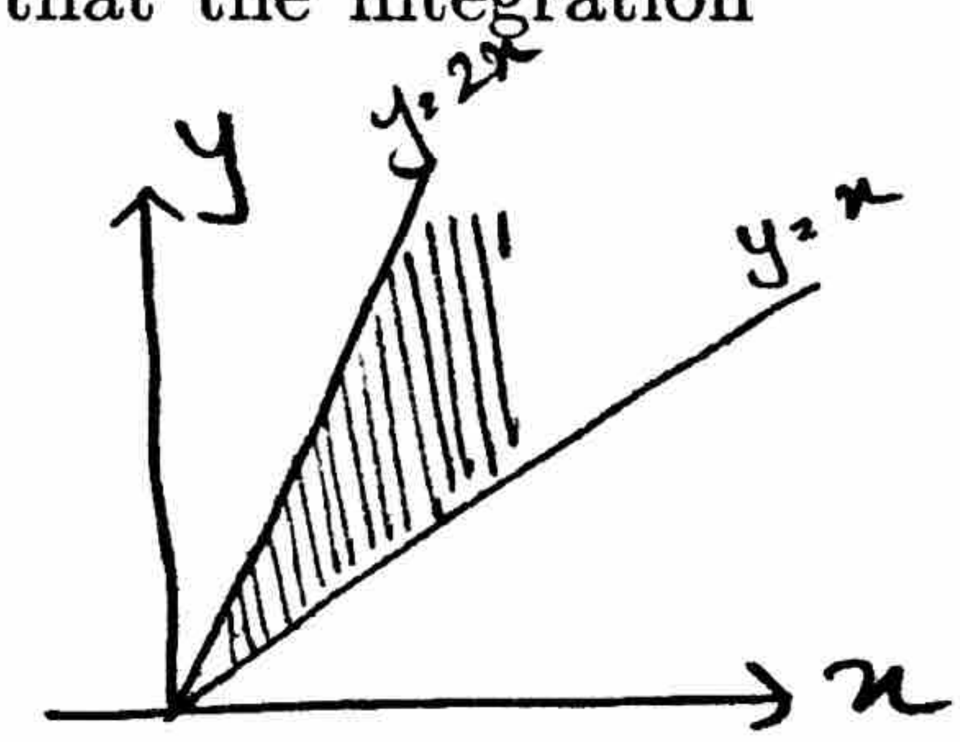
$$f(x,y) = \begin{cases} \frac{2e^{-2x}}{x} & \text{if } 0 \le x \text{ and } x \le y \le 2x \\ 0 & \text{otherwise} \end{cases}$$

Define the expression of

$$F(x,y) = \int_{t=-\infty}^{y} \int_{s=-\infty}^{x} f(s,t)dsdt. \tag{1}$$

1. [2%] Find the value of $F(5, 0)$.
2. [10%] Find the value of $F(4, 10)$.

Hint: You may want to "manipulate" equation (1) slightly so that the integration becomes easier.



$(5,0) = \int_{-\infty}^{0} \int_{-\infty}^{5} f(s,t)\, ds\, dt$

$= 0$, since the function is zero in that region

$F(x,y) = \int_{t=-\infty}^{y} \int_{s=-\infty}^{x} f(s,t)\, ds\, dt$

$= \int_{s=-\infty}^{x} \int_{t=-\infty}^{y} f(s,t)\, dt\, ds$

$F(4,10) = \int_{-\infty}^{4} \int_{-\infty}^{10} f(s,t) dt\, ds$

$= \int_0^4 \int_x^{2x} \frac{2e^{-2x}}{x}\, dy\, dx$

$= \int_0^4 \frac{2e^{-2x}}{x} (2x - x)\, dx$

$= \int_0^4 2e^{-2x}\, dx = -e^{-2x}\Big]_0^4 = -e^{-8} + 1$

$= 1 - e^{-8}$

Question 2: [12%, Work-out question, Learning Objective 1] Consider a discrete function $f(k)$ (k being integer) as follows.

$$f(k) = \begin{cases} 2^{-k} & \text{if } 1 \le k \\ 0 & \text{otherwise} \end{cases}$$

Find the value of the following summation

$$\sum_{k=-\infty}^{\infty} \max(100, k) \cdot f(k) \tag{2}$$

where $\max(100, k)$ returns the maximum of 100 and k. For example, $\max(100, k) = 100$ if $k = 49$ and $\max(100, k) = 787$ if $k = 787$.

Hint: The following formula may be useful.

$$\sum_{n=N}^{\infty} a \cdot n \cdot r^{n-1} = \frac{a \cdot r^{N-1}}{(1-r)^2} + \frac{a \cdot (N-1) \cdot r^{N-1}}{1-r} \tag{3}$$

if $|r| < 1$ and $N \ge 1$.

$$\sum_{k=-\infty}^{\infty} \max(100,k).f(k) = \sum_{k=-\infty}^{99} 100.f(k) + \sum_{k=100}^{\infty} k.f(k)$$

$$= \sum_{k=1}^{99} 100\, 2^{-k} + \sum_{k=100}^{\infty} k\, 2^{-k} = \frac{100}{2} \sum_{k=1}^{99} \frac{1}{2}^{(k-1)} + \frac{1}{2} \sum_{k=100}^{\infty} k \times \frac{1}{2}^{k-1}$$

$$= \frac{100}{2} \left(\frac{1-\left(\frac{1}{2}\right)^{99}}{1-\frac{1}{2}} \right) + \frac{\frac{1}{2}\left(\frac{1}{2}\right)^{99}}{\left(1-\frac{1}{2}\right)^2} + \frac{\frac{1}{2}(99)\left(\frac{1}{2}\right)^{99}}{1-\frac{1}{2}}$$

$$= 100\left(1-\left(\tfrac{1}{2}\right)^{99}\right) + 2\left(\tfrac{1}{2}\right)^{99} + 99\left(\tfrac{1}{2}\right)^{99}$$

$$= 100 - 100\left(\tfrac{1}{2}\right)^{99} + 2\left(\tfrac{1}{2}\right)^{99} + 99\left(\tfrac{1}{2}\right)^{99}$$

$$= 100 + \left(\tfrac{1}{2}\right)^{99}(-100+2+99)$$

$$= 100 + \left(\tfrac{1}{2}\right)^{99}$$

Question 3: [14%, Work-out question, Learning Objective 1] Consider a 1-D continuous function

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We generate another function

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f(x) dx. \tag{5}$$

1. [8%] Find the expression of $M(s)$ assuming $|s| < 2$.
2. [6%] Denote the first order derivative of $M(s)$ by $M'(s)$. Find the value of $M'(0)$.

1) $M(s) = \int_{-\infty}^{\infty} e^{sx} f(x) dx = \int_0^{\infty} e^{sx} 2e^{-2x} dx$

$= \int_0^{\infty} 2e^{(s-2)x} dx = \dfrac{2e^{(s-2)x}}{(s-2)} \Big|_0^{\infty} = \dfrac{2e^{-(2-s)}}{(s-2)} \Big|_0^{\infty} = \dfrac{2}{2-s}$ //

2) $M'(s) = \dfrac{d}{ds} 2(2-s)^{-1} = -2(2-s)^{-2}(-1)$

$= \dfrac{2}{(2-s)^2}$

$M'(0) = \dfrac{2}{2^2} = \dfrac{1}{2}$ //

Question 4: [15%, Work-out question, Learning Objective 1] We consider a continuous random experiment X, which can take any real value. For example, we may have $X = 0.001$.

Suppose the *probability density function* of X is

$$f_X(x) = \begin{cases} c \cdot (1-x) \cdot (-1-x) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

for some constant c. Answer the following questions.

1. [6%] What is the value of c?
2. [9%] What is the probability that "X is larger than 1/3"?

   Hint 1: If you do not know the answer of c, you can still write down your answer by assuming c is a constant. You will receive full credit if your answer is correct.

   Hint 2: Your answer can be something like $17^3 - 8^5$. There is no need to further simplify it.

1) $\int_{-1}^{1} c(1-x)(-1-x)\,dx = -c\int_{-1}^{1} 1^2 - x^2\,dx = 1$

$-c\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 1$

$-c\left[1 - \frac{1}{3} - \left(-1 - \frac{(-1)^3}{3}\right)\right] = 1$

$-c\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right] = 1$

$-c\left[2 - \frac{2}{3}\right] = 1$

$c = -\frac{3}{4}$

2) $P(x > \frac{1}{3}) = \int_{1/3}^{1} \left(-\frac{3}{4}\right)(1-x)(-1-x)\,dx$

$= \frac{3}{4}\int_{1/3}^{1} 1^2 - x^2\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{1/3}^{1}$

$$= \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{1/3}^{1}$$

$$= \frac{3}{4}\left[1 - \frac{1}{3} - \left(\frac{1}{3} - \frac{\left(\frac{1}{3}\right)^3}{3}\right)\right]$$

$$= \frac{3}{4}\left[1 - \frac{2}{3} + \frac{1}{3^4}\right]$$

$$= \frac{3}{4} - \frac{1}{2} + \frac{1}{4 \times 3^3}$$

$$= \frac{1}{4} + \frac{1}{4 \times 3^3}$$

$$= \frac{1}{4}\left(1 + \frac{1}{3^3}\right)$$

$$= \frac{7}{3^3} //$$

Question 5: [16%, Work-out question, Learning Objective 1] Consider three cities A, B, and C with populations 2 millions, 1 million, and 0.5 million, respectively. A salesperson travels through these three cities according to the following rules.

- He spends his first night in city A. That is, he always starts from city A.
- Everyday he will choose a city different from the city he stayed last night. For example, if he stayed in city B last night, he will then randomly choose either city A or city C as his next destination.
- When choosing the city, the probability of choosing that city is *proportional* to the city's population. Continue from the previous example. If he stayed in city B last night, he will choose either city A or city C as his next destination. The probability of choosing A is four times as large as choosing C since the population of city A is four times as large as the population of city C.
- He only travels for 4 days. Namely, he spends 4 nights among these 3 cities.

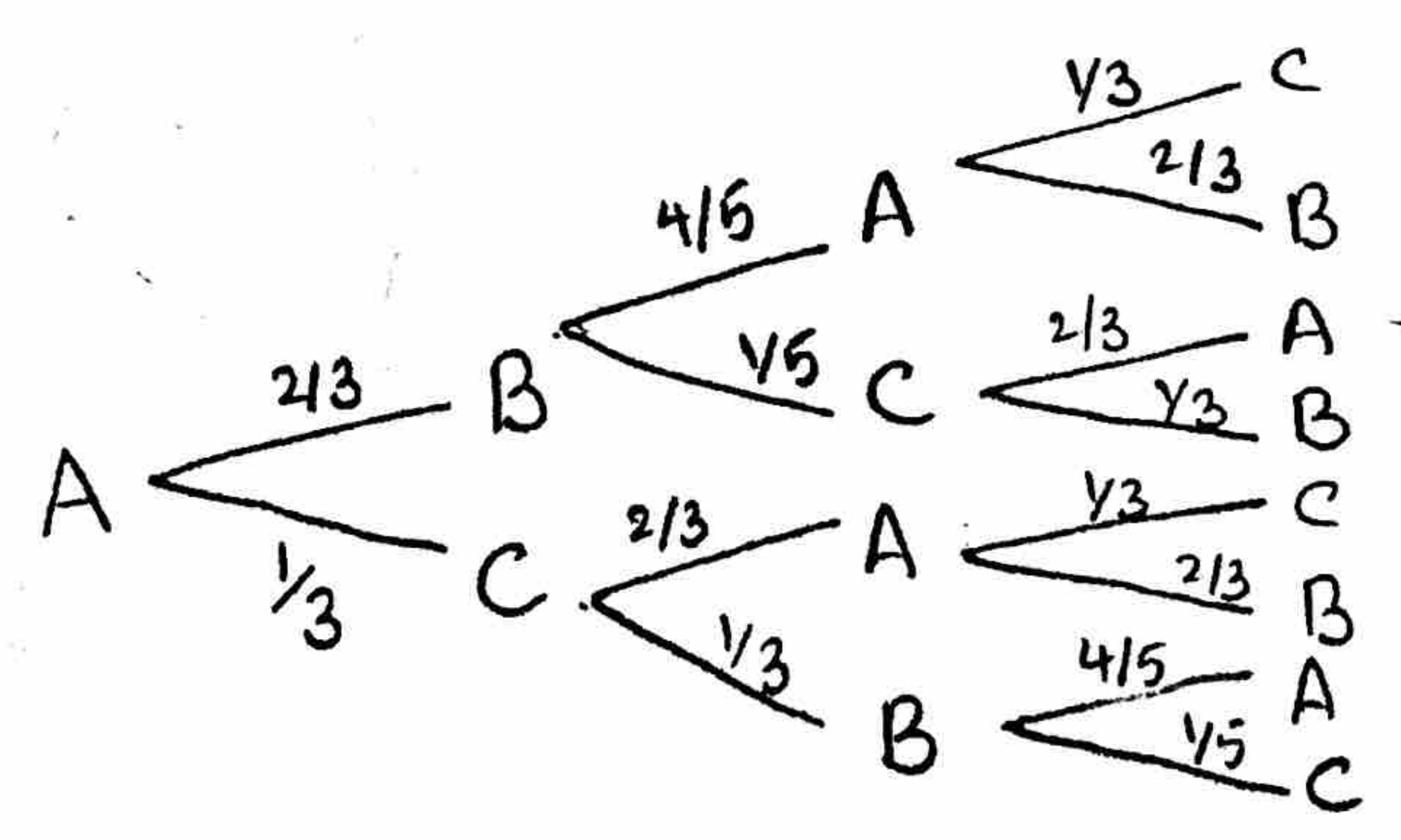
Answer the following questions:

1. [3%] What is the sample space in this experiment?
2. [3%] Use either the tree method or the table method to assign the probabilistic weight.
3. [10%] What is the conditional probability that he has visited all 3 cities given that his fourth night is in city B.

   Hint: Your answer can be something like $\frac{1/56}{1/19+1/29}$. There is no need to further simplify it.

① $S = \{(A,B,A,C),\ (A,B,A,B),\ (A,B,C,A),\ (A,B,C,B),\ (A,C,A,C),\ (A,C,A,B),$
$(A,C,B,A),\ (A,C,B,C)\}$

②



3) $P(\text{visits all 3 cities} \mid \text{night 4 in B})$

$$= \frac{P(\text{visits all 3 cities} \wedge \text{night 4 in city B})}{P(\text{night 4 in city B})}$$

$$= \frac{P(A,B,C,B) + P(A,C,A,B)}{P(A,B,A,B) + P(A,B,C,B) + P(A,C,A,B)}$$

$$= \frac{\frac{2}{3} \times \frac{1}{5} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}}{\frac{2}{3} \times \frac{4}{5} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{5} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}}$$

$$= \frac{\frac{2}{3^2 \times 5} + \frac{4}{3^3}}{\frac{16}{3^2 \times 5} + \frac{2}{3^2 \times 5} + \frac{4}{3^3}} = \frac{2 \times 3 + 4 \times 5}{16 \times 3 + 2 \times 3 + 4 \times 5}$$

$$= \frac{6 + 20}{48 + 6 + 20}$$

$$= \frac{26}{74}$$

$$= \frac{13}{37}$$

Question 6: [21%, Work-out question, Learning Objective 1] Consider a continuous random variable X, which is uniformly randomly chosen from the interval $[1, 6.5]$. (Being continuous means that X can take any values in the range, for example X can be $\sqrt{2}$.)

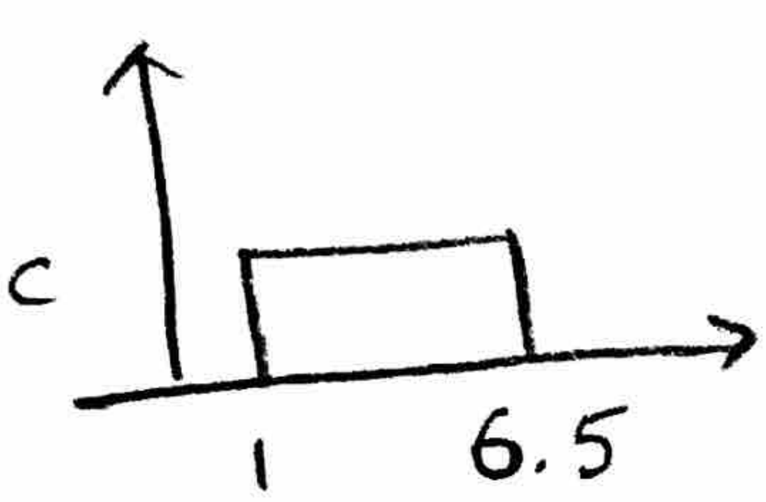
1. [2%] What is the sample space?
2. [4%] What is the probability density function (pdf) you will use to describe the probabilistic weight assignment for X?
3. [7%] Find the probability that $P((X-5)^2 < 3)$. Hint: $\sqrt{3} \approx 1.7$.
4. [8%] We let $Y = \text{round}(X)$. That is Y is the integer to which X is rounded. For example, if $X = \sqrt{2} \approx 1.414$, then $Y = \text{round}(X) = 1$. Another example, if $X = 1.7$, then $Y = \text{round}(X) = 2$. If $X = 2.233$, then $Y = \text{round}(X) = 2$.

   Find the probability $P(Y$ is a prime number$)$.

   Hint: 1 is not a prime number. The smallest prime number is 2.

① $x \in [1, 6.5]$

②



$c = \frac{1}{6.5-1} = \frac{1}{5.5} = \frac{2}{11}$

3) $P((x-5)^2 < 3)$

$P(5-\sqrt{3} < x < 5+\sqrt{3}) = P(3.3 < x < 6.7) = P(3.3 < x < 6.5) = \int_{3.3}^{6.5} \frac{1}{5.5}\,dx = \frac{3.2}{5.5} = \frac{32}{55}$

4) $P(Y \text{ is a prime number}) = \int_{1.5}^{3.5} c\,dx + \int_{4.5}^{5.5} c\,dx$

$= \frac{2}{11}(2) + \frac{2}{11}(1)$

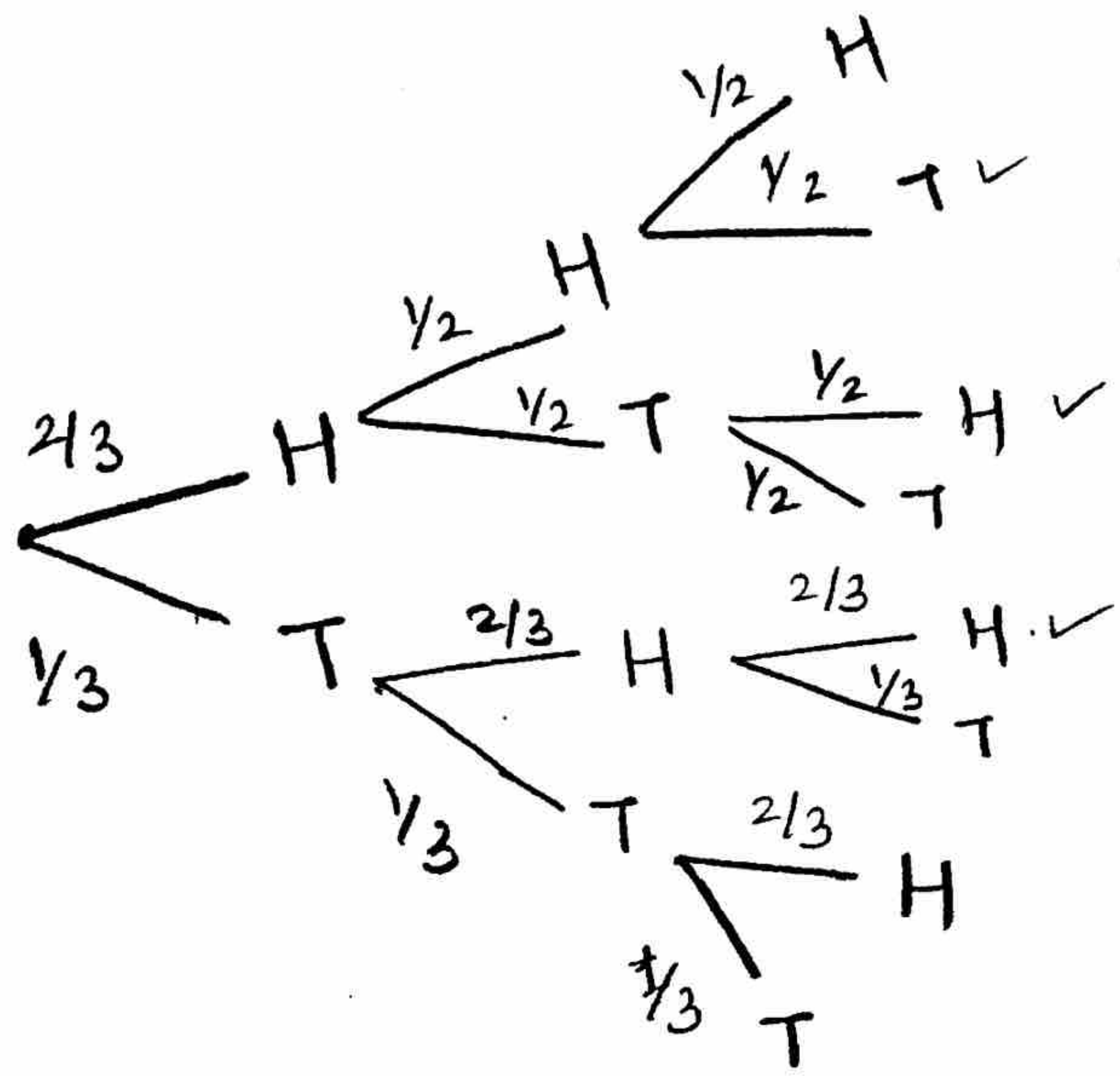
$= \frac{6}{11}$

Question 7: [10%, Work-out question, Learning Objective 1] Consider a bent coin with the head probability being 2/3. Flip the bent coin, if the outcome is head, then flip a different, fair coin twice. If the outcome is tail, then continue flip the bent coin twice.

That is, we always flip a coin three times. Depending on whether the first flip (the bent coin) is head or not, we either flip a fair coin twice or we continue flipping the bent coin twice.

Find the probability that the third flip is a bent coin conditioning on that we observe exactly 2 heads during our 3 flips.

Hint: Your answer can be something like $\frac{1/56}{1/19+1/29}$. There is no need to further simplify it.



P(3rd flip is bent | exactly 2 heads)

$$= \frac{\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}}{\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2}}$$

$$= \frac{4 \times 4}{4 \times 4 + 2 \times 9 + 2 \times 9} = \frac{16}{52} = \frac{4}{13}$$