**Midterm #1 of ECE302, Section 2**

8–9pm, Monday, February 11, 2019, FRNY G140.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam may contain some multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

*Question 1:* [12%, Work-out question, Learning Objective 1] Consider a 2-dimensional continuous function $f(x, y)$.

$$f(x,y) = \begin{cases} \frac{2e^{-2x}}{x} & \text{if } 0 \le x \text{ and } x \le y \le 2x \\ 0 & \text{otherwise} \end{cases}$$

Define the expression of

$$F(x,y) = \int_{t=-\infty}^{y} \int_{s=-\infty}^{x} f(s,t)\,ds\,dt. \tag{1}$$

1. [2%] Find the value of $F(5, 0)$.
2. [10%] Find the value of $F(4, 10)$.

Hint: You may want to "manipulate" equation (1) slightly so that the integration becomes easier.

*Question 2:* [12%, Work-out question, Learning Objective 1] Consider a discrete function $f(k)$ ($k$ being integer) as follows.

$$f(k) = \begin{cases} 2^{-k} & \text{if } 1 \leq k \\ 0 & \text{otherwise} \end{cases}$$

Find the value of the following summation

$$\sum_{k=-\infty}^{\infty} \max(100, k) \cdot f(k) \tag{2}$$

where $\max(100, k)$ returns the maximum of 100 and $k$. For example, $\max(100, k) = 100$ if $k = 49$ and $\max(100, k) = 787$ if $k = 787$.

Hint: The following formula may be useful.

$$\sum_{n=N}^{\infty} a \cdot n \cdot r^{n-1} = \frac{a \cdot r^{N-1}}{(1-r)^2} + \frac{a \cdot (N-1) \cdot r^{N-1}}{1-r} \tag{3}$$

if $|r| < 1$ and $N \geq 1$.

*Question 3:* [14%, Work-out question, Learning Objective 1] Consider a 1-D continuous function

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We generate another function

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f(x) dx. \tag{5}$$

1. [8%] Find the expression of $M(s)$ assuming $|s| < 2$.
2. [6%] Denote the first order derivative of $M(s)$ by $M'(s)$. Find the value of $M'(0)$.

*Question 4:* [15%, Work-out question, Learning Objective 1] We consider a continuous random experiment $X$, which can take any real value. For example, we may have $X = 0.001$.

Suppose the *probability density function* of $X$ is

$$f_X(x) = \begin{cases} c \cdot (1-x) \cdot (-1-x) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

for some constant $c$. Answer the following questions.

1. [6%] What is the value of $c$?

2. [9%] What is the probability that "$X$ is larger than $1/3$"?

   Hint 1: If you do not know the answer of $c$, you can still write down your answer by assuming $c$ is a constant. You will receive full credit if your answer is correct.

   Hint 2: Your answer can be something like $17^3 - 8^5$. There is no need to further simplify it.

*Question 5:* [16%, Work-out question, Learning Objective 1] Consider three cities A, B, and C with populations 2 millions, 1 million, and 0.5 million, respectively. A salesperson travels through these three cities according to the following rules.

- He spends his first night in city A. That is, he always starts from city A.
- Everyday he will choose a city different from the city he stayed last night. For example, if he stayed in city B last night, he will then randomly choose either city A or city C as his next destination.
- When choosing the city, the probability of choosing that city is *proportional* to the city's population. Continue from the previous example. If he stayed in city B last night, he will choose either city A or city C as his next destination. The probability of choosing A is four times as large as choosing C since the population of city A is four times as large as the population of city C.
- He only travels for 4 days. Namely, he spends 4 nights among these 3 cities.

Answer the following questions:

1. [3%] What is the sample space in this experiment?
2. [3%] Use either the tree method or the table method to assign the probabilistic weight.
3. [10%] What is the conditional probability that he has visited all 3 cities given that his fourth night is in city $B$.

   Hint: Your answer can be something like $\frac{1/56}{1/19+1/29}$. There is no need to further simplify it.

*Question 6:* [21%, Work-out question, Learning Objective 1] Consider a continuous random variable $X$, which is uniformly randomly chosen from the interval $[1, 6.5]$. (Being continuous means that $X$ can take any values in the range, for example $X$ can be $\sqrt{2}$.)

1. [2%] What is the sample space?
2. [4%] What is the probability density function (pdf) you will use to describe the probabilistic weight assignment for $X$?
3. [7%] Find the probability that $P((X-5)^2 < 3)$. Hint: $\sqrt{3} \approx 1.7$.
4. [8%] We let $Y = \text{round}(X)$. That is $Y$ is the integer to which $X$ is rounded. For example, if $X = \sqrt{2} \approx 1.414$, then $Y = \text{round}(X) = 1$. Another example, if $X = 1.7$, then $Y = \text{round}(X) = 2$. If $X = 2.233$, then $Y = \text{round}(X) = 2$.

   Find the probability $P(Y \text{ is a prime number})$.

   Hint: 1 is not a prime number. The smallest prime number is 2.

*Question 7:* [10%, Work-out question, Learning Objective 1] Consider a bent coin with the head probability being 2/3. Flip the bent coin, if the outcome is head, then flip a different, fair coin twice. If the outcome is tail, then continue flip the bent coin twice.

That is, we always flip a coin three times. Depending on whether the first flip (the bent coin) is head or not, we either flip a fair coin twice or we continue flipping the bent coin twice.

Find the probability that the third flip is a bent coin conditioning on that we observe exactly 2 heads during our 3 flips.

Hint: Your answer can be something like $\frac{1/56}{1/19+1/29}$. There is no need to further simplify it.

# Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$