Final Exam of ECE302, Section 2

7–9pm, Monday, April 29, 2019, RHPH 172.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains some multiple-choice questions and some work-out questions. For multiple-choice questions, there is no need to justify your answers. You have two hours to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Q1.

$\underline{E(X) = 1.5}$ $\quad E(X^2) = \int_1^2 x^2 \cdot 1 \, dx = \frac{1}{3} x^3 \Big|_1^2 = \frac{7}{3}$

$$Var(X) = \frac{7}{3} - \left(\frac{3}{2}\right)^2 = \frac{28-27}{12} = \underline{\frac{1}{12}}.$$

$$E(Y) = 0.5 \times 1 + 0.5 \times \frac{1}{2} = \underline{\frac{3}{4}}.$$

$$\cancel{Var(} \; E(Y^2) = 0.5 \cdot \int_0^2 y^2 \times 0.5 \, dy + 0.5 \int_0^1 y^2 \times 1 \times dy$$

$$= \frac{1}{4} \times \frac{8}{3} + \frac{1}{2} \cdot \frac{1}{3}$$

$$= \frac{8+2}{12} = \frac{5}{6}.$$

$$Var(Y^2) = \frac{5}{6} - \frac{9}{16} = \frac{40-27}{48} = \frac{13}{48}$$

$$E(XY) = \int_0^2 \int_1^{1.5} xy \times 0.5 \, dx \, dy + \int_0^1 \int_{1.5}^2 xy \, dx \, dy$$

$$= 0.5 \cdot \frac{1}{2}(1.5^2 - 1) \cdot \frac{1}{2} \times 4 + 1 \cdot \frac{1}{2}(4 - 1.5^2) \cdot \frac{1}{2} \times 1$$

$$= \frac{5}{8} + \frac{7}{16}$$

$$= \frac{17}{16}$$

$$\mathrm{Cov}(X,Y) = \frac{17}{16} - \frac{3}{2}\cdot\frac{3}{4} = \frac{-1}{16}$$

$$\rho = \frac{-1/16}{\sqrt{\frac{1}{12}\cdot\frac{13}{48}}} = \frac{-1/16}{\frac{\sqrt{13}}{12\times 2}}$$

$$= \frac{-24}{16\times\sqrt{13}} = \frac{-3}{2\sqrt{13}}$$

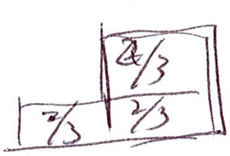
$$E(XY) = \frac{1.25}{2}\times 1 + \frac{1}{4}\cdot(1.75) = \frac{5}{8} + \frac{7}{16}$$

$$= \frac{17}{16}$$

$$\hat{X}_{Lin,MMSE}(y) = \frac{-3}{2\sqrt{13}}\cdot\frac{\sqrt{\frac{1}{12}}}{\sqrt{\frac{13}{48}}}\cdot\left(y-\frac{3}{4}\right)+\frac{3}{2}$$

$$= \frac{-3}{13}\cdot\left(y-\frac{3}{4}\right)+\frac{3}{2}$$

$$\hat{x}_{MMSE}(y) = \begin{cases} \frac{1+1.5}{2} & \text{if } 1 \le y < 2 \\ \int_1^{1.5} \frac{2}{3}\times x\,dx + \int_{1.5}^{2}\frac{4}{3}\times x\times dx & \\ = \frac{1}{3}(1.5^2-1)+\frac{2}{3}(2^2-1.5^2) & \\ = \frac{1}{3}1.25 + \frac{2}{3}\times 1.75 = \frac{5+14}{12} = \frac{19}{12} & \text{if } 0<y<1 \end{cases}$$



$P(X=x|Y=y)$ if $0<y<1$

Q2:

$$E(Y) = E(X_1) + \cdots + E(X_{100})$$

$$= \sum_{i=1}^{100} i \times 0.4$$

$$= \frac{100 \times 101}{2} \times 0.4.$$

$$Var(Y) = \sum_{i=1}^{100} Var(X_i)$$

$$= \sum_{i=1}^{100} i \times 0.4 \times 0.6$$

$$= \frac{100 \times 101}{2} \times 0.4 \times 0.6.$$

Q3:

$m_V = 2 - 1 = 1$

$\sigma_V^2 = 4 \times 1 + \boxed{2 \times 2 \times} (1 \times 2) 0.125 + 4 = 9$

$P(1 + 3Z < 2.5)$

$= P(Z < 0.5) = 1 - Q(0.5) = 1 - 0.3085.$

Q4

$P(X \leq 0.5 \text{ or } Y \leq 0.5)$

$= \frac{0.5}{2} + \frac{0.5}{2} - \left(\frac{0.5}{2}\right)^2$

$= 0.25 + 0.25 - (0.25)^2$

$= \frac{2}{4} - \frac{1}{16} = \frac{7}{16}$

$$P(Z \leq z) = \begin{cases} 0 & \text{if } z < 0 \\ \frac{z}{2} + \frac{z}{2} - \left(\frac{z}{2}\right)^2 & 0 \leq z < 2 \\ 1 & 2 \leq z \end{cases}$$

$$f_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ 1 - \frac{z}{2} & \text{if } 0 \leq z < 2 \\ 0 & 2 \leq z \end{cases}$$

Q5:

$$f_{XY}(x,y) = \begin{cases} 2 \cdot e^{-2x} \cdot \frac{1}{x} \cdot e^{-\frac{1}{x} y} & \text{if } 0<x,\ 0<y \\ 0 & \text{otherwise} \end{cases}$$

$$E(Y) = E(E(Y|X))$$
$$= E(X) = \frac{1}{2}$$

Q6.

1. $P(X+Y=4)$

$$= \binom{5}{4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^1$$

$$= 5 \times \frac{\times 2}{3^5} = \frac{10}{3^5} = \frac{10}{243}$$

2. $P(Z\cancel{0}=1) = 0.5 \times 0.3 + 0.5 \times 0.7 = 0.5$

$$P(Z=1 \mid Y=1) = \frac{P(Z=1 \cancel{0}, Y=1)}{P(Y=1)}$$

$$= \frac{\frac{1}{3} \cdot 0.5}{\frac{1}{3}} = 0.5$$

Independent.

Q7:

1. $E(X^2) = Var(X) + (m_X)^2$

$$= \frac{1-p}{p^2} + \left(\frac{1-p}{p}\right)^2$$

$$= \frac{1-0.2}{0.2^2} + \left(\frac{1-0.2}{0.2}\right)^2$$

$$= \frac{0.80}{0.04} + (4)^2$$

$$= 36$$

2. $$= \frac{P(X=3)}{P(0 \le X \le 4)}$$

$$= \frac{0.2 \times 0.8^3}{0.2 \cdot 0.8^0 + 0.2 \times 0.8^1 + 0.2 \times 0.8^2 + 0.2 \times 0.8^3 + 0.2 \times 0.8^4}$$

Question 8: [18%, True/false question. There is no need to justify your answers]
Decide whether the following statements are true or false.

True 1. [2%] X and Y are uniformly distributed in a unit circle centered at the $(-2, 0)$, i.e., those (x, y) satisfying $(x+2)^2 + y^2 \leq 1$. The random variables X and Y are orthogonal.

False 2. [2%] Consider three random variables X, Y, and Z. Suppose X and Y have correlation coefficient $\rho_{XY} = -1$, and Y and Z have correlation coefficient $\rho_{YZ} = 1$. Then X and Z have correlation coefficient $\rho_{XZ} = 0$.

True 3. [2%] X is a standard Gaussian random variable, Y is Bernoulli with parameter $p = 0.5$. Define $Z = X \times (2Y - 1)$. Then Z is standard Gaussian random variable.

False 4. [2%] X is a standard Gaussian random variable, Y is Bernoulli with parameter $p = 0.5$. Define $Z = X \times (2Y - 1)$. Then $W = X + Z$ is a Gaussian random variable.

True 5. [2%] Both X and Y are strictly positive random variables satisfying $P(X > 0.1) = 1$ and $P(Y > 0.2) = 1$. We also know that X and Y are *orthogonal*. Then X and Y must be negatively correlated.

There was some error when designing Q8.5. Please ignore this question when studying.

False 6. [2%] Let $F_{Y|X}(y|x)$ to be the conditional cdf of random variable Y given X and let $F_Y(y)$ be the conditional cdf of Y without the conditioning. We must have $0 \leq F_{Y|X}(y|x) \leq F_Y(y)$.

False 7. [2%] X, Y, and Z are standard Gaussian random variables and they are joinlty independent. Define their avarge as $W = \frac{X+Y+Z}{3}$. Then W is also a standard Gaussian random variable.

False 8. [2%] X_1 is a Gaussian random variable with $m_1 = 1$, $\sigma_1^2 = 1$; and X_2 is a Gaussian random variable with $m_2 = 2$, $\sigma_2^2 = 2$. Then for any value x, we must have $P(X_1 \geq x) \leq P(X_2 \geq x)$.

True 9. [2%] X_1 is a Poisson random variable with $\alpha_1 = 3$ and X_2 is a Poisson random variable with $\alpha_2 = 4$. Then for any value x, we must have $P(X_1 \geq x) \leq P(X_2 \geq x)$.