Final Exam of ECE302, Section 2

7–9pm, Monday, April 29, 2019, RHPH 172.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains some multiple-choice questions and some work-out questions. For multiple-choice questions, there is no need to justify your answers. You have two hours to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [20%, Work-out question, Learning Objectives 1 and 2] Consider two random variables X and Y with the following joint pdf function.

$$f_{X,Y}(x,y) = \begin{cases} 0.5 & \text{if } 1 < x < 1.5 \text{ and } 0 < y < 2 \\ 1 & \text{if } 1.5 \leq x < 2 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

1. [12%] Find the linear MMSE estimator of X given Y.

   Hint 1: If you do not know how to solve this question, you can find the following values instead: $E(X)$, $E(Y)$, $\mathsf{Var}(X)$, $\mathsf{Var}(Y)$, and $\rho_{X,Y}$ instead. You will receive 9 points if your answers are correct.

2. [8%] Find the MMSE estimator of X given Y and plot $\hat{X}_{\text{MMSE}}(y)$ for the range of $0 \leq y \leq 2$.

   Hint 2: If you do not know the answer of this sub-question, you can write down what does the acronym MMSE stand for, and also write down what is the formula for computing the MMSE estimator. If your answer is correct, you will receive 4 points.

   Hint 3: This question is relatively more challenging. You may want to work on other questions first.

Question 2: [10%, Work-out question, Learning Objectives 1 and 2]

Consider 100 random variables X_1 to X_{100}. We know that for all $i = 1$ to 100 X_i is a binomial random variable (R.V.) with parameter $n = i$ and $p = 0.4$. For example, X_4 is a binomial R.V. with parameter $n = 4$ and $p = 0.4$ and X_{76} is a binomial R.V. with $n = 76$ and $p = 0.4$. We also know that these 100 random variables are *independent*.

We define $Y = X_1 + X_2 + \cdots + X_{100}$. Answer the following questions.

1. [5%] Find the value of $E(Y)$.

   Hint 1: If you do not know the answer to this question, you can assume $W = X_1 + X_2$ and find the value of $E(W)$ instead. You will receive 2 points if your answer is correct.

   Hint 2: Your answer must not contain $\sum$. However you do not need to write down the simplest value and your answer can be of the form of $\frac{98 \times 172 \times 3}{2}$.

2. [5%] Find the value of $\mathsf{Var}(Y)$.

   Hint 3: If you do not know the answer to this subquestion, write down the expression of $\mathsf{Var}(Y)$ in terms of $\mathsf{Var}(X_1)$ to $\mathsf{Var}(X_{100})$. You will receive 3.5 points if your answer is correct.

   Hint 4: Your answer must not contain $\sum$. However you do not need to write down the simplest value and your answer can be of the form of $\frac{98 \times 172 \times 3}{2}$.

Question 3: [9%, Work-out question, Learning Objectives 1 and 2] X and Y are joint Gaussian random variables with $m_X = 1$, $\sigma_X^2 = 1$, $m_Y = -1$, $\sigma_Y^2 = 4$ and $\rho_{XY} = 1/8$. Answer the following questions.

1. [6%] Define a new random variable $U = 2X + Y$. Find the mean m_U and variance σ_U^2 of U.

2. [3%] Find the probability that $P(2X + Y < 2.5)$?

Hint 1: The following values may be useful.

$$Q(0) = 0.5 \tag{2}$$
$$Q(0.25) = 0.4013 \tag{3}$$
$$Q(0.5) = 0.3085 \tag{4}$$
$$Q(0.75) = 0.2266 \tag{5}$$

Hint 2: You do not need to write down the simplest value. Instead, your answer can be of the form of $1 - 0.3723 + 0.5133$.

Question 4: [12%, Work-out question, Learning Objectives 1 and 2] Consider two random variables X and Y. X is uniformly distributed over $(0, 2)$, Y is uniformly distributed over $(0, 2)$, and X and Y are independent. Define $Z = \min(X, Y)$.

1. [8%] Find the probability $P(Z \leq 0.5)$.

   Hint 1: The statement "$\min(X, Y) \leq 0.5$" is equivalent to "$X \leq 0.5$ or $Y \leq 0.5$".

   Hint 2: If you do not know the answer to this sub-question, please find the probabilities: $P(X \leq 0.5)$, $P(Y \leq 0.5)$, and $P(X \leq 0.5$ and $Y \leq 0.5)$ instead. You will receive 6 points if your answer is correct.

2. [4%] Find the pdf of Z.

   Hint 2: You may want to find the cdf of Z first.

Question 5: [10%, Work-out question, Learning Objectives 1 and 2] Consider two random variables X and Y. X is exponentially distributed with parameter $\lambda_X = 2$. Given $X = x_0$, Y is exponentially distributed with $\lambda_Y = \frac{1}{x_0}$.

1. [3%] Write down the joint pdf $f_{X,Y}(x, y)$.
2. [7%] Find the expectation $E(Y)$.

   Hint 1: If you do not know the answer of this subquestion, you can find the value of $E((X-3)^2)$ instead. You will receive 4 points if your answer is correct.

   Hint 2: The formula tables in the end of this exam booklet may be useful.

Question 6: [13%, Work-out question, Learning Objectives 1 and 2]

1. [8%] Consider two independent random variables X and Y. Specifically X is a binomial random variable with $n = 2$ and $p = \frac{1}{3}$, and Y is a binomial distribution with $n = 3$ and $p = \frac{1}{3}$. Find the probability of $P(X + Y = 4)$.

2. [5%] Consider two independent random variables X and Y. Specifically X is a Bernoulli random variable with $p = 0.5$, and Y is a Bernoulli random variable with $p = \frac{1}{3}$. Define $Z = X \oplus Y$. Namely, Z is the binary-exclusive-or (XOR) of X and Y.

   (A more detailed definition is that if $X = 0$ and $Y = 0$, then $Z = 0$; if $X = 0$ and $Y = 1$, then $Z = 1$; if $X = 1$ and $Y = 0$, then $Z = 1$; and if $X = 1$ and $Y = 1$, then $Z = 0$.)

   Answer the following question: Are the events $\{Y = 1\}$ and $\{Z = 1\}$ independent?

   Hint: This is NOT a yes-no question. Please carefully write down your justification.

Question 7: [8%, Work-out question, Learning Objectives 1 and 2] Consider a geometric random variable X with parameter $p = 0.2$.

1. [4%] Find the second moment of X.
2. [4%] Find the conditional probability $P(X = 3|(X-2)^2 < 5.5)$.

Question 8: [18%, True/false question. There is no need to justify your answers]

Decide whether the following statements are true or false.

1. [2%] X and Y are uniformly distributed in a unit circle centered at the $(-2, 0)$, i.e., those (x, y) satisfying $(x + 2)^2 + y^2 \leq 1$. The random variables X and Y are orthogonal.

2. [2%] Consider three random variables X, Y, and Z. Suppose X and Y have correlation coefficient $\rho_{XY} = -1$, and Y and Z have correlation coefficient $\rho_{YZ} = 1$. Then X and Z have correlation coefficient $\rho_{XZ} = 0$.

3. [2%] X is a standard Gaussian random variable, Y is Bernoulli with parameter $p = 0.5$, and X and Y are independent. Define $Z = X \times (2Y - 1)$. Then Z is standard Gaussian random variable.

4. [2%] X is a standard Gaussian random variable, Y is Bernoulli with parameter $p = 0.5$, and X and Y are independent. Define $Z = X \times (2Y - 1)$. Then $W = X + Z$ is a Gaussian random variable.

5. [2%] Both X and Y are strictly positive random variables satisfying $P(X > 0.1) = 1$ and $P(Y > 0.2) = 1$. We also know that X and Y are *orthogonal*. Then X and Y must be negatively correlated.

6. [2%] Let $F_{Y|X}(y|x)$ to be the conditional cdf of random variable Y given X and let $F_Y(y)$ be the conditional cdf of Y without the conditioning. We must have $0 \leq F_{Y|X}(y|x) \leq F_Y(y)$.

7. [2%] X, Y, and Z are standard Gaussian random variables and they are joinlty independent. Define their avarge as $W = \frac{X+Y+Z}{3}$. Then W is also a standard Gaussian random variable.

8. [2%] X_1 is a Gaussian random variable with $m_1 = 1$, $\sigma_1^2 = 1$; and X_2 is a Gaussian random variable with $m_2 = 2$, $\sigma_2^2 = 2$. Then for any value x, we must have $P(X_1 \geq x) \leq P(X_2 \geq x)$.

9. [2%] X_1 is a Poisson random variable with $\alpha_1 = 3$ and X_2 is a Poisson random variable with $\alpha_2 = 4$. Then for any value x, we must have $P(X_1 \geq x) \leq P(X_2 \geq x)$.

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\text{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\text{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\text{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\text{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega} - 1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \leq x \leq b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.

- 2-dimensional Gaussian Random Vector

  $S = \{(x, y) : \text{for all real-valued } x \text{ and } y\}$

  $$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)}$$

  $E(X) = m_X$, $\mathrm{Var}(X) = \sigma_X^2$, $E(Y) = m_Y$, $\mathrm{Var}(Y) = \sigma_Y^2$, and $\mathrm{Cov}(X, Y) = \rho\sqrt{\sigma_X^2\sigma_Y^2}$.

- n-dimensional Gaussian Random Variable

  $S = \{(x_1, x_2, \cdots, x_n) : \text{for all real-valued } x_1 \text{ to } x_n\}$

  If we denote $\vec{x} = (x_1, x_2, \cdots, x_n)$ as an n-dimensional row-vector, then the pdf of an n-dimensional Gaussian random vector becomes

  $$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(K)}} e^{-\frac{1}{2}(\vec{x}-\vec{m})K^{-1}(\vec{x}-\vec{m})^{\mathrm{T}}}$$

  where $\vec{m}$ is the mean vector of X, i.e., $\vec{m} = E(\vec{X})$; K is an $n \times n$ covariance matrix, where the (i, j)-th entry of the K matrix is $\mathrm{Cov}(X_i, X_j)$; $\det(K)$ is the determinant of K; and K^{-1} is the inverse of K.