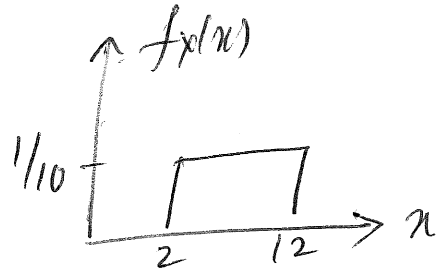
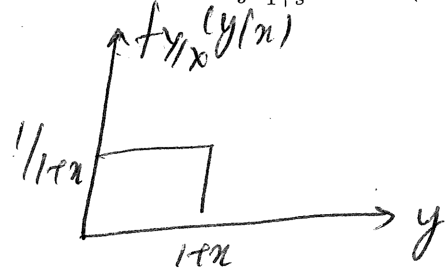
Question 1: [25%, Work-out question] Suppose the continuous random variable X is uniformly distributed between with $[2, 12]$. Given $X = x_0$, the continuous random variable Y is uniformly distributed between $[0, 1 + x_0]$.

1. [12%] Find the expectation $E(XY)$.
2. [13%] Let $F_{XY}(x, y)$ denote the joint cdf of X and Y. Find the expression of the $F_{XY}(x, y)$ when assuming $x < 12$. Namely, you do not need to compute the expression of $F_{XY}(x, y)$ when $12 \leq x$.

   Hint 1: If you do not know how to discuss the cases, you can find out the values of $F_{XY}(10, 7)$, $F_{XY}(7, 10)$, separately. You will receive 10 points if you compute both values correctly.

   Hint 2: The following equation may be useful: $\int \frac{1}{1+s} ds = \ln(1 + s)$.





$$E[XY] = \iint xy\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_2^{12} f_X(x) \left(\int_0^{1+x} xy\, f_{Y/X}(y/x)\, dy \right) dx$$

$$= \int_2^{12} \frac{1}{10} \int_0^{1+x} y \frac{1}{1+x}\, dy\, dx = \int_2^{12} \frac{1}{10} \left(\frac{x}{1+x}\right)\left(\frac{y^2}{2}\right)\Big|_0^{1+x} dx$$

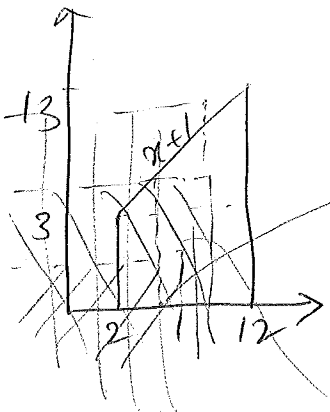
$$= \frac{1}{10} \int_2^{12} \frac{x(1+x)}{2}\, dx = \frac{1}{10} \left(\frac{12^2 - 2^2}{4} + \left(\frac{12^3 - 2^3}{6}\right)\right)$$

$$= \frac{1}{10} (35 + 286.67) = 32.16$$

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{x} f_X(x) \int_{-\infty}^{y} f_{Y/X}(y/x)\,dy\,dx$$

- $f_X(x) = 0 \quad x < 2$
  $f_{Y/X}(y/x) = 0 \quad y < 0$ $\Rightarrow F_{XY}(x,y) = 0$



- $x < 12 \Rightarrow f_X(x) = 1/10$

$y < 3 \Rightarrow$

$$F_{XY}(x,y) = \int_{2}^{x} \frac{1}{10} \int_{0}^{y} \frac{1}{1+x}\,dy\,dx$$

$$= \frac{y}{10} Ln\left[\frac{(x+1)}{3}\right]$$

$y < x+1$

$$F_{XY}(x,y) = \int_{2}^{y-1} \frac{1}{10} \int_{0}^{x+1} \frac{1}{1+x}\,dy\,dx + \int_{y-1}^{x} \frac{1}{10} \int_{0}^{y} \frac{1}{1+x}\,dy\,dx$$

$$= \frac{(y-3)}{10} + \frac{y}{10} Ln\left(\frac{x+1}{y}\right)$$

$y > x+1$

$$F_{XY}(x,y) = \int_{2}^{x} \frac{1}{10} \int_{0}^{1+x} \frac{1}{1+x}\,dy\,dx$$

$$= \left(\frac{x-2}{10}\right)$$

Question 2: [20%, Work-out question] Consider a geometric random variable X with parameter $p = \frac{2}{3}$. We can use X to generate another random variable $Y = 2X + 1$.

1. [10%] Find the expression of the probability generating function $G_Y(z)$ of Y.

   Hint 1: If you do not know how to find the probability generating function, you can find the following expectation $E(e^{-sY})$ instead. You will receive 8 points if your answer is correct.

2. [10%] Use the probability generating function $G_Y(z)$ to find the mean of Y.

   Hint 1: If you do not know the answer to the previous sub-question, you can assume that $G_Y(z) = (\frac{2+z}{3})^{20}$. You will still get 10 points if your answer is correct.

   Hint 2: If you do not know how to answer this question, you can use any other method to solve the mean of Y. You will receive 6 points if your answer is correct.

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$$P(X=k) = (1-p)^k p \qquad k \in \{0,1,\ldots \infty\}, \quad p \to 2/3$$

$$G_x(z) = E[z^x]$$

$$= \sum_{k=0}^{\infty} z^k (1-p)^k p = p \sum_{k=0}^{\infty} (z(1-p))^k$$

$$= \frac{p}{1-(1-p)z}$$

$$G_y(z) = E[z^y]$$

$$= E[z^{2x+1}] = zE[(z^2)^x] = \frac{zp}{1-(1-p)z^2}$$

$$\mu_y = \left.\frac{dG_y(z)}{dz}\right|_{z=1}$$

$$= \frac{(1-(1-p)z^2)p - zp(0-(1-p)2z)}{(1-(1-p)z^2)^2}$$

$$= \frac{p^2 + p(1-p)2}{(1-(1-p))^2} = \frac{p^2+2p-2p^2}{p^2} = \frac{2-p}{p}$$

$$= \frac{2-2/3}{2/3}$$

$$= \underline{\underline{2}}$$

$$G_y(z) = \left(\frac{2+z}{3}\right)^{20}$$

$$\Rightarrow \mu_y(z) = \left. 20\left(\frac{2+z}{3}\right)^{19} \frac{1}{3}\right|_{z=1}$$

$$= 20/3$$

Question 3: [20%, Work-out question] We know that X is a binomial distribution with parameter $n = 10000$ and $p = \frac{1}{2}$. We use X to create another random variable $Y = \frac{X}{10000}$. That is, X is the total number of heads after flipping a fair coin 10000 times and Y is the *empirical frequency* (normalized by the trials) after the coin flipping.

1. [5%] Express the probability $P(Y \geq \frac{2}{3})$ as a summation.

   Hint: You do not need to expand/compute the value of the summation. Something like $\sum_k \frac{1}{3^k}$ would suffice.

2. [10%] Use the Chebyshev inequality to upper bound the probability $P(|Y - 0.5| \geq \frac{1}{6})$.

3. [5%] Write down how to use the Chernoff inequality to upper bound the probability $P(Y \geq \frac{2}{3})$.

   Hint 1: You do not need to compute the Chernoff inequality in this sub-question. Instead, all you need to do is to write down "what is the Chernoff inequality" explicitly and then provide step-by-step description how you plan to compute the Chernoff inequality.

   Hint 2: The moment generating function of a binomial distribution is $X^*(s) = (1 - p + pe^{-s})^n$.

4. [Bonus 5%] Compute the Chernoff inequality value exactly, so that you can upper bound the probability $P(Y \geq \frac{2}{3})$.

   Hint 3: This is a bonus question. So even if you do not answer this question, you can still get 20 points if your answers to the previous sub-questions are correct.

   Hint 4: You may want to use the fact that $e^{s \cdot 5}(3 + s)^{10} = (e^{\frac{s \cdot 5}{10}} \cdot (3 + s))^{10}$.

$$P(Y \geq 2/3) = P\left(X \geq \frac{20000}{3}\right)$$

$$= \sum_{k=6667}^{10000} {}^{10000}C_k \left(1/2\right)^n$$

$$P(|Y - 0.5| \geq 1/6) \leq \frac{\sigma_Y^2}{(1/6)^2}$$

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2 \rightarrow 1/2$$

$$\left(\frac{E[X^2]}{10^8} = \frac{\sigma_X^2 + \mu_X^2}{10^8} = \frac{1}{4}\left(\frac{10^4 + 10^8}{10^8}\right)\right)$$

$$= 10^{-4}/4$$

$$\Rightarrow P(|Y - 0.5| \geq 1/6) \leq 9 \times 10^{-4}$$

$$P(X \geq a) \leq \frac{E[e^{-tx}]}{e^{-ta}}$$

$$P(Y \geq 2/3) = P\left(X \geq \frac{20000}{3}\right) \leq e^{t\left(\frac{20000}{3}\right)} E[e^{-tX}]$$

$$\leq e^{t\left(\frac{20000}{3}\right)} (1-p+pe^{-t})^{10000}$$

$$\leq e^{t\left(\frac{20000}{3}\right)} \left(\frac{1}{2}+\frac{1}{2}e^{-t}\right)^{10000}$$

$$\leq \left(e^{\frac{2t}{3}} \left(\frac{1}{2} + \frac{1}{2} e^{-t} \right) \right)^{10000}$$

$$\leq \left(\frac{1}{2} e^{\frac{2t}{3}} + \frac{1}{2} e^{\frac{-t}{3}} \right)^{10000}$$

Bound : $\min_{t \leq 0} \left(\frac{1}{2} e^{\frac{2t}{3}} + \frac{1}{2} e^{\frac{-t}{3}} \right)^{10000}$

$\Rightarrow \min_{t \leq 0} \left(\frac{1}{2} e^{2t/3} + 1/2\, e^{-t/3} \right)$

$$\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} e^{\frac{2t}{3}} + \frac{1}{2} \cdot \frac{-1}{3} e^{-t/3} = 0$$

$$\frac{e^{\frac{2t}{3}}}{3} - \frac{1}{6} e^{-t/3} = 0$$

$$\Rightarrow \frac{e^{\frac{2t}{3}}}{\cancel{3}} = \frac{e^{-t/3}}{\cancel{6}\, 2}$$

$$\Rightarrow e^{t} = 1/2$$

$$\Rightarrow \boxed{t = -\ln 2}$$

$\therefore$ Bound : $\left(\frac{1}{2} e^{\frac{-2\ln 2}{3}} + \frac{1}{2} e^{\frac{\ln 2}{3}} \right)^{10000}$ //

Question 4: [15%, Work-out question] The continuous random variable Y is uniformly distributed on $(1, 4)$; the continuous random variable X is exponentially distributed with parameter $\lambda = 1$; and X and Y are independent. Let $Z = X + Y$. Find the expectation of $E(Z^2)$.

$$E(Z^2) = E(X+Y)^2$$

Independent

$$= E(X^2) + E(Y^2) + 2E(X)E(Y)$$

$$= Var(X) + E(X)^2 + Var(Y) + \mu_Y^2 + 2\mu_X\mu_Y$$

$$= \lambda^{-2} + \lambda^{-2} + \frac{1}{12}(4-1)^2 + \left(\frac{5}{2}\right)^2 + 2 \cdot \frac{5}{2}\lambda^{-1}$$

$$= 1 + 1 + \frac{\cancel{9}\,3}{\cancel{12}\,4} + \frac{25}{4} + 5$$

$$= 14$$

Question 5: [20%, Multiple choice question. There is no need to justify your answers]

YES 1. [2%] X and Y are Bernoulli distributed with parameters $p_X = 0.5$ and $p_Y = 0.5$, respectively. Compute $Z = X \oplus Y$, i.e., Z is the binary exclusive or of X and Y. For example $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, and $1 \oplus 1 = 0$. Are X and Z independent?

yes 2. [2%] X and Y are Bernoulli distributed with parameters $p_X = 0.5$ and $p_Y = \frac{1}{3}$, respectively. Compute $Z = X \oplus Y$. Are Y and Z independent?

no 3. [2%] X and Y are Bernoulli distributed with parameters $p_X = 0.5$ and $p_Y = \frac{1}{3}$, respectively. Compute $Z = X \oplus Y$. Are X and Z independent?

YES 4. [2%] Consider a random variable X with mean $m = 4$ and variance $\sigma^2 = 25$. Is the following inequality always true: $P((X - m) \leq -10) \leq \frac{\sigma^2}{100}$?

no 5. [2%] Suppose we know that $P(X \geq Y) = 1$. Is the following inequality always true: $F_X(100.7) \geq F_Y(100.7)$ where $F_X(x)$ and $F_Y(y)$ are the marginal cdf of X and Y respectively?

NO 6. [3%] Suppose X is Gaussian with $\mu = 1$ and $\sigma^2 = 4$ and $Y = -X + 1$. Is it true that $P(|Y| < 3) = 1 - Q(1.5)$.

$-Q(0.5)$

YES 7. [3%] Is the following statement always true? "If two random variables X and Y are independent, then we have $E(e^{3X+2Y}) = (E(e^{3X}))(E(e^{2Y}))$."

NO 8. [2%] Is the following statement true? "X is a binomial random variable with $n = 4$ and $p = 0.5$. Given $X = x_0$, the conditional probability of Y is Poisson with parameter $\alpha = x_0$. In this scenario, X and Y are independent."

yes 9. [2%] Is the following statement always true? "If X is binomial distribution with $n = 100$ and $p = 1$; and Y is also binomial distributed with the same parameter value $n = 100$ and $p = 1$, then X and Y are independent."