Midterm #3 of ECE302, Section 2

8–9pm, Wednesday, April 5, 2017, FRNY G140.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [25%, Work-out question] Suppose the continuous random variable X is uniformly distributed between with $[2, 12]$. Given $X = x_0$, the continuous random variable Y is uniformly distributed between $[0, 1 + x_0]$.

1. [12%] Find the expectation $E(XY)$.

2. [13%] Let $F_{XY}(x, y)$ denote the joint cdf of X and Y. Find the expression of the $F_{XY}(x, y)$ when assuming $x < 12$. Namely, you do not need to compute the expression of $F_{XY}(x, y)$ when $12 \leq x$.

   Hint 1: If you do not know how to discuss the cases, you can find out the values of $F_{XY}(10, 7)$, $F_{XY}(7, 10)$, separately. You will receive 10 points if you compute both values correctly.

   Hint 2: The following equation may be useful: $\int \frac{1}{1+s} ds = \ln(1 + s)$.

Question 2: [20%, Work-out question] Consider a geometric random variable X with parameter $p = \frac{2}{3}$. We can use X to generate another random variable $Y = 2X + 1$.

1. [10%] Find the expression of the probability generating function $G_Y(z)$ of Y.

   Hint 1: If you do not know how to find the probability generating function, you can find the following expectation $E(e^{-sY})$ instead. You will receive 8 points if your answer is correct.

2. [10%] Use the probability generating function $G_Y(z)$ to find the mean of Y.

   Hint 1: If you do not know the answer to the previous sub-question, you can assume that $G_Y(z) = (\frac{2+z}{3})^{20}$. You will still get 10 points if your answer is correct.

   Hint 2: If you do not know how to answer this question, you can use any other method to solve the mean of Y. You will receive 6 points if your answer is correct.

Question 3: [20%, Work-out question] We know that X is a binomial distribution with parameter $n = 10000$ and $p = \frac{1}{2}$. We use X to create another random variable $Y = \frac{X}{10000}$. That is, X is the total number of heads after flipping a fair coin 10000 times and Y is the *empirical frequency* (normalized by the trials) after the coin flipping.

1. [5%] Express the probability $P(Y \geq \frac{2}{3})$ as a summation.

   Hint: You do not need to expand/compute the value of the summation. Something like $\sum_k \frac{1}{3^k}$ would suffice.

2. [10%] Use the Chebyshev inequality to upper bound the probability $P(|Y - 0.5| \geq \frac{1}{6})$.

3. [5%] Write down how to use the Chernoff inequality to upper bound the probability $P(Y \geq \frac{2}{3})$.

   Hint 1: You do not need to compute the Chernoff inequality in this sub-question. Instead, all you need to do is to write down "what is the Chernoff inequality" explicitly and then provide step-by-step description how you plan to compute the Chernoff inequality.

   Hint 2: The moment generating function of a binomial distribution is $X^*(s) = (1 - p + pe^{-s})^n$.

4. [Bonus 5%] Compute the Chernoff inequality value exactly, so that you can upper bound the probability $P(Y \geq \frac{2}{3})$.

   Hint 3: This is a bonus question. So even if you do not answer this question, you can still get 20 points if your answers to the previous sub-questions are correct.

   Hint 4: You may want to use the fact that $e^{s \cdot 5}(3 + s)^{10} = (e^{\frac{s \cdot 5}{10}} \cdot (3 + s))^{10}$.

Question 4: [15%, Work-out question] The continuous random variable Y is uniformly distributed on $(1, 4)$; the continuous random variable X is exponentially distributed with parameter $\lambda = 1$; and X and Y are independent. Let $Z = X + Y$. Find the expectation of $E(Z^2)$.

Question 5: [20%, Multiple choice question. There is no need to justify your answers]

1. [2%] X and Y are Bernoulli distributed with parameters $p_X = 0.5$ and $p_Y = 0.5$, respectively. Compute $Z = X \oplus Y$, i.e., Z is the binary exclusive or of X and Y. For example $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, and $1 \oplus 1 = 0$. Are X and Z independent?

2. [2%] X and Y are Bernoulli distributed with parameters $p_X = 0.5$ and $p_Y = \frac{1}{3}$, respectively. Compute $Z = X \oplus Y$. Are Y and Z independent?

3. [2%] X and Y are Bernoulli distributed with parameters $p_X = 0.5$ and $p_Y = \frac{1}{3}$, respectively. Compute $Z = X \oplus Y$. Are X and Z independent?

4. [2%] Consider a random variable X with mean $m = 4$ and variance $\sigma^2 = 25$. Is the following inequality always true: $P((X - m) \leq -10) \leq \frac{\sigma^2}{100}$?

5. [2%] Suppose we know that $P(X \geq Y) = 1$. Is the following inequality always true: $F_X(100.7) \geq F_Y(100.7)$ where $F_X(x)$ and $F_Y(y)$ are the marginal cdf of X and Y respectively?

6. [3%] Suppose X is Gaussian with $\mu = 1$ and $\sigma^2 = 4$ and $Y = -X + 1$. Is it true that $P(|Y| < 3) = 1 - Q(1.5)$.

7. [3%] Is the following statement always true? "If two random variables X and Y are independent, then we have $E(e^{3X+2Y}) = (E(e^{3X}))(E(e^{2Y}))$."

8. [2%] Is the following statement true? "X is a binomial random variable with $n = 4$ and $p = 0.5$. Given $X = x_0$, the conditional probability of Y is Poisson with parameter $\alpha = x_0$. In this scenario, X and Y are independent."

9. [2%] Is the following statement always true? "If X is binomial distribution with $n = 100$ and $p = 1$; and Y is also binomial distributed with the same parameter value $n = 100$ and $p = 1$, then X and Y are independent."

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\mathrm{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\mathrm{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega}-1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \le x \le b$.

  $E(X) = \frac{a+b}{2}$, $\text{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\text{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\text{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\text{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.

- 2-dimensional Gaussian Random Vector

  $S = \{(x, y) : \text{for all real-valued } x \text{ and } y\}$

  $$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)}$$

  $E(X) = m_X$, $\text{Var}(X) = \sigma_X^2$, $E(Y) = m_Y$, $\text{Var}(Y) = \sigma_Y^2$, and $\text{Cov}(X, Y) = \rho\sqrt{\sigma_X^2\sigma_Y^2}$.

- n-dimensional Gaussian Random Variable

  $S = \{(x_1, x_2, \cdots, x_n) : \text{for all real-valued } x_1 \text{ to } x_n\}$

  If we denote $\vec{x} = (x_1, x_2, \cdots, x_n)$ as an n-dimensional row-vector, then the pdf of an n-dimensional Gaussian random vector becomes

  $$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(K)}} e^{-\frac{1}{2}(\vec{x}-\vec{m})K^{-1}(\vec{x}-\vec{m})^{\mathrm{T}}}$$

  where $\vec{m}$ is the mean vector of X, i.e., $\vec{m} = E(\vec{X})$; K is an $n \times n$ covariance matrix, where the (i, j)-th entry of the K matrix is $\text{Cov}(X_i, X_j)$; $\det(K)$ is the determinant of K; and K^{-1} is the inverse of K.