Midterm #2 of ECE302, Section 2

8–9pm, Tuesday, February 28, 2017, FRNY G140.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID: SOLUTION

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [25%, Work-out question, Learning Objective 1]

1. [6%] X is a uniform random variable with $(a, b) = (\sqrt{2}, 4)$. Find the *third central moment* of X.

   Hint: If you do not know what is the third central moment, you can find the $E(X)$ instead. You will still receive 3 points if your answer is correct.

2. [6%] X is a geometric random variable with $p = \frac{3}{4}$. Find the conditional probability $P(2^X \leq 10 | X^2 \leq 40.5)$.

3. [8%] X is a binomial random variable with $(n, p) = (40, \frac{1}{3})$. Suppose $Y = -2X + 1$ find the values of $E(Y)$, $E(Y^2)$, and $\mathsf{Var}(Y)$.

4. [5%] X is an exponential random variable with $\lambda = 2$. Find the probability $P(|X - 2| < 5)$.

1. $X \sim \text{Unif}(a,b) = \text{Unif}(\sqrt{2},4)$

$E((X-\mu)^3) = \int_{\sqrt{2}}^{4} (x-\mu)^3 \frac{dx}{(4-\sqrt{2})}$ $\quad x - \mu = z \Rightarrow x = z+\mu \;\&\; dx = dz$

$\mu = \frac{4+\sqrt{2}}{2}$

$= \frac{1}{4-\sqrt{2}} \int_{\frac{\sqrt{2}-4}{2}}^{\frac{4-\sqrt{2}}{2}} z^3 dz = 0 \longrightarrow$ Skewness is zero for a Uniform RV

$\hookrightarrow$ odd function

2. Geometric RV $\sim p(1-p)^{k-1} \rightsquigarrow p, p(1-p), p(1-p)^2 \ldots$ $\quad k \in \{1, 2, \ldots \infty\}$

$p(1-p)^k \rightsquigarrow p, p(1-p), p(1-p)^2 \ldots$ $\quad k \in \{0, 1, \ldots \infty\}$

$P(2^X \leq 10 / X^2 \leq 40.5)$

$\hookrightarrow X \leq \log_2 10 \sim 3.3$ $\quad \hookrightarrow > 36, < 49$

$= P(X < 4 / X < 7)$ or $P(X \leq 3 / X \leq 6)$

$= \frac{P(X \in \{0,1,2,3\})}{P(X \in \{0,1 \ldots 6\})} = \frac{3/4 + 3/4 \cdot 1/4 + 3/4(1/4)^2}{3/4 + 3/4 \cdot 1/4 + 3/4(1/4)^2 \ldots \frac{3}{4}(1/4)^5}$

$$= \frac{(1+1/4 \cdots (1/4)^3)}{(1+1/4 \cdots (1/4)^6)}$$

$$= \frac{(1/4)^4 - 1}{(1/4)^7 - 1}$$

3. $Y = -2X + 1$ $\to Np$

$\Rightarrow E[Y] = -2E[X] + 1$

$= -2 \times \frac{40}{3} + 1 = -25.67$ or $-\frac{77}{3}$

$Y^2 = 4X^2 + 1 - 4X$

$\Rightarrow E[Y^2] = 4E[X^2] + 1 - 4E[X]$

$\left(\begin{array}{l} Var(X) = Np(1-p) = 40 \times \frac{1}{3} \times \frac{2}{3} = \frac{80}{9} \\ \Rightarrow E[X^2] = Var(X) + (E(X))^2 = \frac{80}{9} + \left(\frac{40}{3}\right)^2 = \frac{80}{9} + \frac{1600}{9} = \frac{1680}{9} \end{array} \right)$

$= \frac{6720}{9} + 1 - \frac{160}{3} = \frac{6720 + 9 - 480}{9} = \frac{6249}{9}$

$Var(Y) = 4\,Var(X) = \frac{320}{9}$

4. $P(|X-2| < 5) = P(-3 < X < 7) = P(0 < X < 7)$ or $P(X < 7)$

$= \int_0^7 2e^{-2x}\,dx = \cancel{2}\,\frac{e^{-2x}}{-\cancel{2}}\Big|_0^7$

$= 1 - e^{-14}$

Question 2: [16%, Work-out question, Learning Objective 1]

1. [2%] What does the acronym "pdf" stand for?

X and Y are two random variables with the sample space of (X, Y) being
2/9 1/9 3/9 1/9 1/9 1/9
$\{(1,0), (1,1), (2,0), (2,1), (3,0), (3,1)\}$. That is, X can be chosen from 1, 2, and 3 and Y can be chosen from 0 and 1.

The weight assignment of (X, Y) is as follows.

$$P(X = x, Y = y) = \begin{cases} \frac{2}{9} & \text{if } x = 1, y = 0 \\ \frac{3}{9} & \text{if } x = 2, y = 0 \\ \frac{1}{9} & \text{all other 4 cases} \end{cases} \tag{1}$$

Define the following function

$$f(x) = \begin{cases} 1 & \text{if } x < \sqrt{2} \\ 2 & \text{if } x \geq \sqrt{2} \end{cases}, \tag{2}$$

and we construct another random variable Z by setting $Z = f(X)$.

2. [7%] Are the two random variables X and Y *independent*? This is NOT a yes/no question. You need to carefully justify your answer. An answer without justification will receive 0 point in this sub-question.

3. [7%] Are the two random variables Z and Y *independent*? This is NOT a yes/no question. You need to carefully justify your answer. An answer without justification will receive 0 point in this sub-question.

   Hint 1: $\sqrt{2} \approx 1.4$.

   Hint 2: If you do not know the answer to this question, you can instead solve the weight assignment of (Z, Y). You will receive 5 points if your answer is correct.

1. Probability Density function

3. $Z = f(X) \Rightarrow X = 1, 2, 3$
$Z = 1, 2$

2/9 1/9 4/9 2/9
$\Rightarrow (Z, Y) = \{(1,0), (1,1), (2,0), (2,1)\}$

$P(Y=0) = 2/3$, $P(Y=1) = 1/3$, $P(Z=1) = 1/3$, $P(Z=2) = 2/3$

$\Rightarrow$ Z and Y are Independent. (Can be verified for each case...)

2. $P(X=1) = 1/3$

$P(X=2) = 4/9$

$P(X=3) = 2/9$

$P(Y=0) = 2/3$

$P(Y=1) = 1/3$

X and Y are <u>not</u> Independent

as $P_{XY}(2,0) \neq P_X(2)\, P_Y(0)$

$3/9 \neq 4/9 \cdot 2/3$

Question 3: [22%, Work-out question, Learning Objective 1] We flip a bent coin and and denote the outcome by Y. (We always use 1 for Head and 0 for Tail.) Since the coin is bent, we assume that it is Bernoulli distributed with $p = 0.25$. Depending on the Y value, computer generates a continuous random variable X that is uniformly distributed between $2Y$ and $2Y+2$. For example, if $Y = 1$, then X is uniformly distributed between 2 and 4.

1. [14%] Find the expression of the cdf of X, denoted by $F_X(x)$, and plot $F_X(x)$ versus x for the range of $-5 < x < 5$.

2. [8%] Find the pdf of X, denoted by $f_X(x)$, and plot it for the range of $-5 < x < 5$.

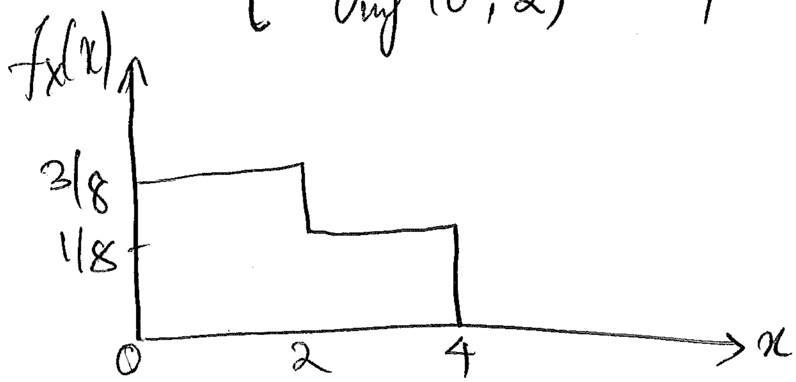
   Hint: If you do not know the answer to the previous subquestion, you can assume

$$F_X(x) = \begin{cases} 0 & \text{if } x < 3 \\ \frac{x^2}{25} & \text{if } 3 \le x < 4 \\ 1 - \frac{9}{(x+1)^2} & \text{if } 4 \le x \end{cases} \tag{3}$$

   You will still get full credit (7 pts) if your answer is correct.

$$Y \begin{cases} 1 & p = 0.25 \\ 0 & p = 0.75 \end{cases}$$

$$X = f(Y) \begin{cases} \text{Unif } (2,4) & p = 0.25 \\ \text{Unif } (0,2) & p = 0.75 \end{cases}$$

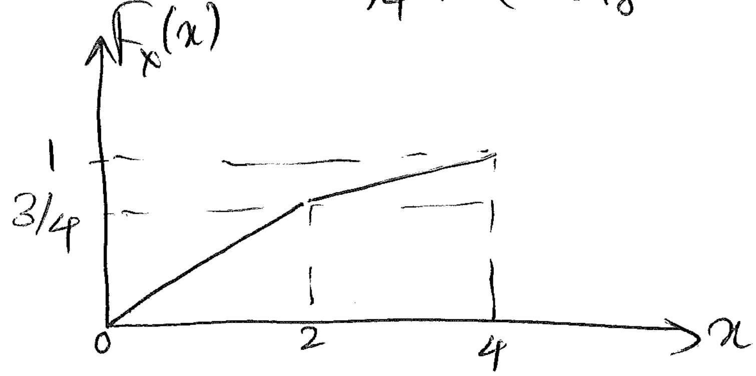


$$F_X(x) = \int_0^x f_X(x)\,dx = \int_0^x 3/8\,dx \qquad x \in (0,2)$$

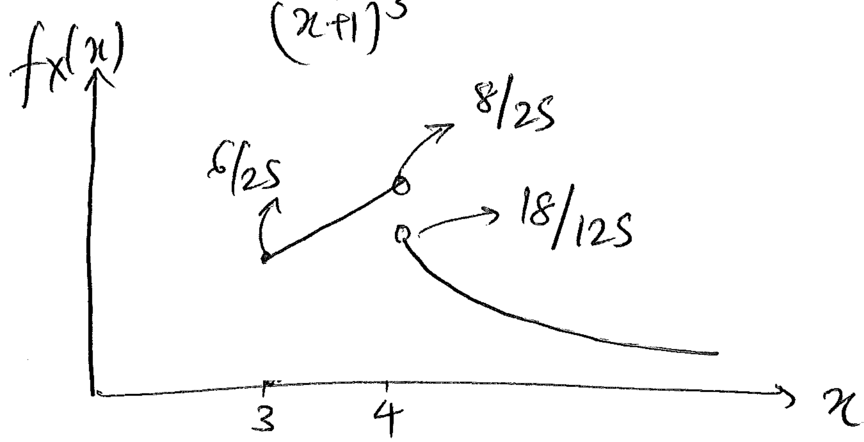
$$= \frac{3x}{8}$$

$$= \frac{3}{4} + \int_2^x \frac{1}{8}\,dx = \frac{3}{4} + \frac{x-2}{8} \qquad x \in (2,4)$$

$$\Rightarrow F_X(x) \begin{cases} \frac{3}{8}x & x \in [0,2) \\ \frac{3}{4} + (x-2)/8 & x \in [2,4] \end{cases}$$



$$f_X(x) \begin{cases} 0 & x < 3 \\ \frac{2x}{25} & x \in [3,4) \\ \frac{18}{(x+1)^3} & x \geq 4 \end{cases}$$



Question 4: [22%, Work-out question, Learning Objective 1] A cell phone company is trying to find out the root cause of the battery explosion of its cell phones. Here is what they have found thus far.

Factory A has very high quality control. In average, there are 0.5 defective batteries coming out of Factory A every day, i.e., the defective rate is 0.5 batteries per day. Factory B has very poor quality control and in average there are 10 defective batteries coming out of Factory B every day, i.e., the defective rate is 10 batteries per day.

Both factories started making batteries 10 days ago. Fortunately, the quality control at the headquarter found out that Factory B's poor quality 5 days ago and stopped its production ever since. (Namely, Factory B only produced batteries for 5 days while Factory A continues producing batteries for 10 days.)

Assume that the numbers of defective batteries are Poisson distributed. Answer the following questions.

1. [8%] What is the probability that Factory B has produced ≥ 10 defective batteries?

2. [10%] Country X has imported all its cell phones/batteries from a single Factory (either A or B) but we do not know which factory is supplying country X. There is a 50/50 chance that the supplier is Factory A or Factory B. However, the newspapers have already reported 10 defective batteries in country X. What is the conditional probability that the supplier of country X is Factory A, given that there are already 10 incidents?

   Hint: You do not need to compute the final number. A carefully written sum/interation formula that explains your idea should suffice.

3. [4%] Obviously the more incidents being reported, the more one can be sure that the supplier is actually Factory B. Suppose there are D defective batteries that have been reported. How large the D value needs to be until you can be 99% sure that the supplier of country X is Factory B? Explain how you plan to find the value of D.

   Hint: A carefully outlined procedure how you plan to find the value of D would suffice.

1. $\lambda_B = 10$ batteries per day

$= 50$ batteries per 5 days

$$P(X \geq 10) = \sum_{k=10}^{\infty} \frac{e^{-50}(50)^k}{k!}$$

2. $\lambda_A = 5$ batteries per 10 days

$$P(X=10) = \left.\frac{e^{-5}5^k}{k!}\right|_{k=10} \text{ if from factory A } = \frac{e^{-5}5^{10}}{10!}$$

$$P(X=10) = \left.\frac{e^{-50}50^k}{k!}\right|_{k=10} \quad \text{if from factory B}$$

$$= \frac{e^{-50}50^{10}}{10!}$$

$$\Rightarrow P(X=10) = \frac{1}{2}\left(\frac{e^{-50}50^{10}}{10!} + \frac{e^{-5}5^{10}}{10!}\right)$$

$$P(X=10 \text{ from } A \,/\, X=10) = \frac{\frac{e^{-5}5^{10}}{\cancel{10!}}}{\frac{e^{-5}5^{10}}{\cancel{10!}} + \frac{e^{-50}50^{10}}{\cancel{10!}}}$$

3. $P(X=D \text{ from } D \,/\, X=D)$

$$= \frac{e^{-50}50^D}{e^{-50}50^D + e^{-5}5^D} = \frac{99}{100}$$

$$\Rightarrow \frac{e^{-50}50^D}{e^{-5}5^D} = 99$$

$$\Rightarrow e^{-45}10^D = 99$$

$$\Rightarrow D = \log_{10}(99e^{+45})$$

Question 5: [15%, Work-out question, Learning Objective 1]

X is an exponential random variable with parameter $\lambda = 2$. Define

$$Y = \min(5, 2X) \tag{4}$$

Find the expectation $E(Y)$.

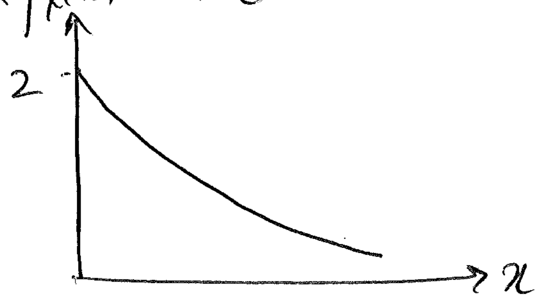
Hint 1: You may want to find the cdf of Y first.

Hint 2: There is a way to directly use the cdf of Y to find $E(Y)$.

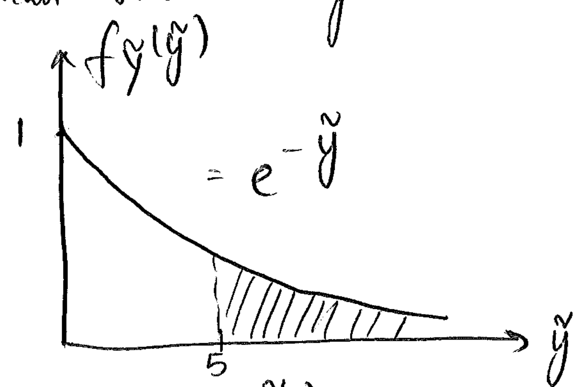
Hint 3: Alternatively, you can also find $E(Y)$ by finding the pdf of Y.

Hint 4: This question is one of the harder questions. You may want to solve this question after you have answered the other questions.

$f_X(x) = 2e^{-2x}$



Consider the transformation $\quad \tilde{y} = 2X \Rightarrow f_{\tilde{Y}}(\tilde{y}) = \frac{2e^{-2x}}{d\tilde{y}/dx} = e^{-\tilde{y}}$



$Y = \min(5, \tilde{Y})$

$$\int_5^\infty e^{-y}\,dy = \frac{e^{-y}}{-1}\Big|_5^\infty = e^{-5}$$

$$\Rightarrow f_Y(y) = (u(x) - u(x-5))e^{-y} + e^{-5}\delta(y-5)$$

$$\Rightarrow E[Y] = \int_0^\infty y f_Y(y)\,dy = \int_0^5 y e^{-y}\,dy + 5e^{-5}$$

$$= y\frac{e^{-y}}{-1}\Big|_0^5 + \int_0^5 e^{-y}\,dy + 5e^{-5} = \cancel{-5e^{-5}} + \frac{e^{-y}}{-1}\Big|_0^5 + \cancel{5e^{-5}} = (1 - e^{-5})$$

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\mathrm{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\mathrm{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega} - 1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \leq x \leq b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.

- 2-dimensional Gaussian Random Vector

  $S = \{(x, y) : \text{for all real-valued } x \text{ and } y\}$

  $$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)}$$

  $E(X) = m_X$, $\mathrm{Var}(X) = \sigma_X^2$, $E(Y) = m_Y$, $\mathrm{Var}(Y) = \sigma_Y^2$, and $\mathrm{Cov}(X, Y) = \rho\sqrt{\sigma_X^2\sigma_Y^2}$.

- n-dimensional Gaussian Random Variable

  $S = \{(x_1, x_2, \cdots, x_n) : \text{for all real-valued } x_1 \text{ to } x_n\}$

  If we denote $\vec{x} = (x_1, x_2, \cdots, x_n)$ as an n-dimensional row-vector, then the pdf of an n-dimensional Gaussian random vector becomes

  $$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(K)}} e^{-\frac{1}{2}(\vec{x}-\vec{m})K^{-1}(\vec{x}-\vec{m})^{\mathrm{T}}}$$

  where $\vec{m}$ is the mean vector of X, i.e., $\vec{m} = E(\vec{X})$; K is an $n \times n$ covariance matrix, where the (i, j)-th entry of the K matrix is $\mathrm{Cov}(X_i, X_j)$; $\det(K)$ is the determinant of K; and K^{-1} is the inverse of K.