Midterm #1 of ECE302, Section 2

8–9pm, Wednesday, February 01, 2017, FRNY G140.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID: SOLUTION

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [12%, Work-out question, Learning Objective 1] Consider a dimensional continuous function $f(x,y)$.

$$f(x,y) = \begin{cases} \frac{xe^{-xy}}{3} & \text{if } 1 \le x \le 4 \text{ and } 0 \le y \\ 0 & \text{otherwise} \end{cases}$$

Define the expression of

$$M(s,t) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} e^{s\cdot x + t\cdot y} f(x,y)\,dx\,dy. \tag{1}$$

1. [12%] Find the expression of $M(0,t)$ assuming $t < 1$.

Hint: You may want to "manipulate" equation (1) slightly so that the integration becomes easier.

$$M(s,t) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} e^{(sx+ty)} \frac{xe^{-xy}}{3}\,dx\,dy$$

$$M(0,t) = \frac{1}{3} \int_{y=0}^{\infty} \int_{x=1}^{4} e^{ty}\, x e^{-xy}\,dx\,dy$$

$$= \frac{1}{3} \int_{x=1}^{4} x \left(\int_{y=0}^{\infty} e^{(t-x)y}\,dy \right) dx$$

$$= \frac{1}{3} \int_{1}^{4} x \left. \frac{e^{(t-x)y}}{(t-x)} \right|_{y=0}^{\infty} dx$$

$t < 1$
$x \geq 1$
$\Rightarrow t - x < 0$

$$= \frac{1}{3} \int_{1}^{4} x \frac{-1}{(t-x)}\,dx = \frac{1}{3} \int_{1}^{4} \frac{x}{x-t}\,dx$$

Substituting $x - t = z \Rightarrow x = z + t$ & $dx = dz$

$$= \frac{1}{3} \int_{1-t}^{4-t} \frac{z+t}{z}\,dz = \frac{1}{3} \int_{1-t}^{4-t} \left(1 + \frac{t}{z}\right) dz$$

$$= \frac{1}{3}\int_{1-t}^{4-t}$$

$$= \frac{1}{3}\left(z + t\ln z \Big|_{1-t}^{4-t}\right)$$

$$= \frac{1}{3}\left(3 + t\ln\frac{4-t}{1-t}\right)$$

$$= 1 + \frac{t}{3}\ln\left(\frac{4-t}{1-t}\right) \;//$$

Question 2: [10%, Work-out question, Learning Objective 1] Consider a discrete function $f(k)$ (k being integer) as follows.

$$f(k) = \begin{cases} \frac{2}{3^{76}} \cdot 3^{-k} & \text{if } 75 \leq k \rightarrow k \geq 75 \\ 0 & \text{otherwise} \end{cases}$$

Find the expression of the following function, assuming $|z| < 3$.

$$g(z) = \sum_{k=-\infty}^{\infty} z^k f(k). \tag{2}$$

$$g(z) = \sum_{75}^{\infty} z^k \frac{2}{3^{76}} \cdot 3^{-k}$$

$$= \frac{2}{3^{76}} \sum_{75}^{\infty} \left(\frac{z}{3}\right)^k$$

$$= \frac{2}{3^{76}} \frac{(z/3)^{75}}{1 - z/3} \quad \text{as } |z| < 3$$

$$= \left(\frac{6}{3^{151}}\right) \frac{z^{75}}{3 - z} = \left(\frac{2}{3^{150}}\right) \frac{z^{75}}{3 - z}$$

Question 3: [18%, Work-out question, Learning Objective 1] Consider a 1-D function.

$$f(x) = \begin{cases} x^2 & \text{if } -1 \le x < 1 \\ \frac{1}{3}e^{-x+1} & \text{if } 1 \le x \\ 0 & \text{otherwise} \end{cases}. \tag{3}$$

We generate another function

$$F(x) = \int_{s=-\infty}^{x} f(s)ds. \tag{4}$$

Find the expression of $F(x)$ and plot it for the range of $-3 \le x \le 3$.

Hint: If you do not know how to proceed, you can compute the following three values instead: $F(-2)$, $F(0.5)$, $F(2)$. If your answers of these three values are correct, you will still receive 14 points.

$F(-2) = 0$
$F(0.5) = 0.375$
$F(2) = 0.88$

- $F(x) = 0 \quad x < -1$ as $f(x) = 0$
- $x \in [-1, 1]$

$$F(x) = \int_{-1}^{x} x^2 dx = \left.\frac{x^3}{3}\right|_{-1}^{x} = \frac{(x^3+1)}{3}$$

$$F(1) = 2/3$$

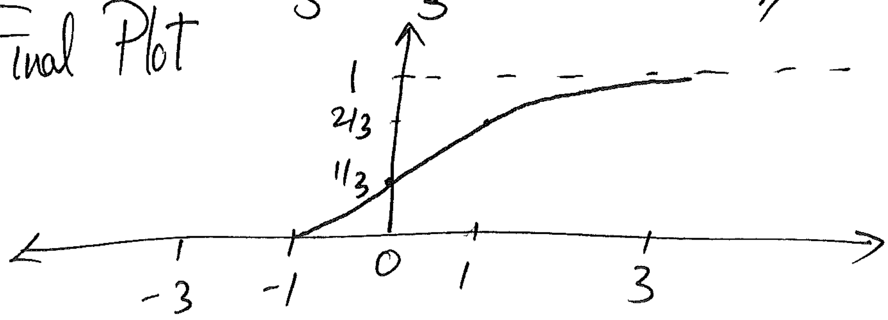
- $x \in [1, \infty)$

$$F(x) = \frac{2}{3} + \int_{1}^{x} \frac{1}{3}(e^{-x}) e \, dx$$

$$= \frac{2}{3} + \frac{e}{3} \left.\frac{e^{-x}}{-1}\right|_{1}^{x} = \frac{2}{3} + \frac{e}{3}(e^{-1} - e^{-x})$$

$$= \frac{2}{3} + \frac{1}{3}(1 - e^{1-x}) \rightarrow F(\infty) = 1$$

- Final Plot



Question 4: [17%, Work-out question, Learning Objective 1] We know that the next-day Fedex delivery service guarantees package delivery between 8am to 12pm. Let X denote the actual delivery time of my package. For example if $X = 8.5$, it means that the package is delivered at 8:30am; if $X = 9 + \frac{5}{6}$, it means that the package is delivered at 9:50am; and so on so forth.

Suppose the *probability density function* of X is

$$f_X(x) = \begin{cases} c(x-8) & \text{if } 8 \le x \le 10 \\ 2c & \text{if } 10 < x \le 11 \\ 2c(12-x) & \text{if } 11 < x \le 12 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

for some constant c. Answer the following questions.

1. [6%] What is the value of c?
2. [9%] Suppose I have to run some errands and am away during 10:30am and 11:30am. What is the probability that "the delivery guy arrives after 11am" or "I miss the delivery (since I am away during 10:30-11:30am)".

   Hint: If you do not know the answer of c, you can still write down your answer by assuming c is a constant. You will receive full credit if your answer is correct.

1. $\int_x f_X(x)\,dx = 1$

$$\Rightarrow \int_8^{10} c(x-8)\,dx + \int_{10}^{11} 2c\,dx + \int_{11}^{12} 2c(12-x)\,dx = 1$$

$$c\left(\frac{x^2}{2} - 8x\right)\Big|_8^{10} + 2cx\Big|_{10}^{11} + 2c\left(12x - \frac{x^2}{2}\right)\Big|_{11}^{12} = 1$$

$$c(18-16) + (2c)1 + 2c\left(12 - \frac{23}{2}\right) = 1$$

$$2c + 2c + c = 1 \Rightarrow c = 1/5 \,//$$

2. ~~P(Missing the delivery i.e delivery guy arrives after 11am~~)

$= P$(delivery guy arrives between 11am to 1130am) . . . ①

$$= \int_{11}^{11.5} 2c(12-x)\,dx = \frac{2}{5}\int_{11}^{11.5}(12-x)\,dx = \frac{2}{5}\left(12x - \frac{x^2}{2}\right)\Big|_{11}^{11.5}$$

$$= \frac{2}{5}\left(12 \times 0.5 - \frac{1}{2}(22.5)(0.5)\right)$$

$$= \frac{2}{5}(6 - 5.625)$$

$$= .15 \quad = 15\%$$

P(delivery guy arrives between 10:30 am to 11am) ....(2)

$$= \int_{10.5}^{11} 2c \, dx = 2cx \Big|_{10.5}^{11}$$

$$= 2c \times 0.5 = c = \frac{1}{5} = 20\%$$

⇒ ~~Probability of missing the delivery if the delivery guy~~

P(Missing the delivery) $= P(10.5 < X < 11.5) = 35\%$ $\begin{pmatrix} 15+20 \\ (1) + (2) \end{pmatrix}$

P(delivery guy arriving after 11am) $= P(X > 11)$ ....(3)

$$= \frac{2}{5}\int_{11}^{12} 12 - x \, dx = \frac{2}{5}\left\{\left(12x - \frac{x^2}{2}\right)\right\}\Big|_{11}^{12} = \frac{2}{5}(12 - 11.5)$$

$$= 20\%$$

P(delivery guy arriving after 11am ∪ missing the delivery)

$$= P(\{X > 11\} \cup \{10.5 < X < 11.5\})$$

$$= P(\{10.5 < X < 12\})$$

$$= (2) + (3)$$

$$= 40\%$$

Question 5: [15%, Work-out question, Learning Objective 1] Prof. Wang uses the following way to deciding where/how he would like to spend his spring break. He has 2 fair dice, die #1 and die #2. He tossed two dice simultaneously and records its outcome as (X_1, X_2), where X_1 is the outcome of die #1 and X_2 is the outcome of die #2.

If $X_1 + X_2 \geq 10$, then he will travel to Hawaii for his spring break. If $|X_1 - X_2|$ is a prime number (the smallest prime number is 2), he will buy a new iPad and read e-books during the spring break. (He may purchase an iPad and fly to Hawaii if both conditions are satisfied simultaneously).

Answer the following questions:

1. [3%] What is the sample space in this experiment?

2. [3%] Use either the tree method or the table method to assign the probabilistic weight.

3. [12%] If you see Prof. Wang using a new iPad after spring break, what is the conditional probability he spent his spring break in Hawaii, given that he has purchased a new iPad.

1. (1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
   (2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
   (3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
   (4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
   (5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)
   (6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)

2. 36 events, equally probable with weights (1/36)

3. $X_1 + X_2 \geq 10 \rightarrow (4,6)\ (6,4)\ (5,6)\ (6,5)\ (6,6) \rightarrow E_1$

   $|X_1 - X_2| \rightarrow$ prime (circled in 1) $\rightsquigarrow$ 16 outcomes $\rightarrow E_2$

   $P(E_1/E_2) = \dfrac{P(E_1 \cap E_2)}{P(E_2)} = \dfrac{2}{16} = \dfrac{1}{8}$

Question 6: [14%, Work-out question, Learning Objective 1] Consider a continuous random variable X, which is uniformly randomly chosen from the interval $[1.2, 2.4]$. (Being continuous means that X can take any values in the range, for example X can be $\sqrt{3}$.)

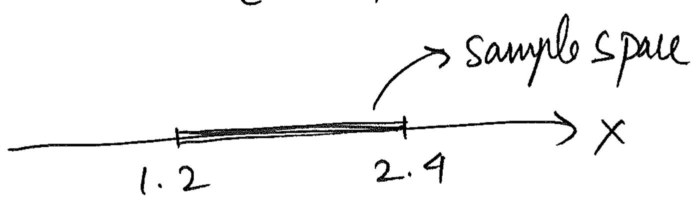
1. [2%] What is the sample space?
2. [4%] What is the probability density function (pdf) you will use to describe the probabilistic weight assignment for X?
3. [8%] Find the probability that $P\left(\cos\left(\frac{\pi \cdot X}{4.8}\right) < 0.5\right)$.

   Hint 1: You may want to plot the function $g(x) = \cos\left(\frac{\pi \cdot x}{4.8}\right)$, which will help you understand what does the above probability mean. You should pay special attention to the value of $g(1.2)$, $g(1.8)$, and $g(2.4)$. This will help you understand what kind of X values needs to be "counted" when computing the probability.

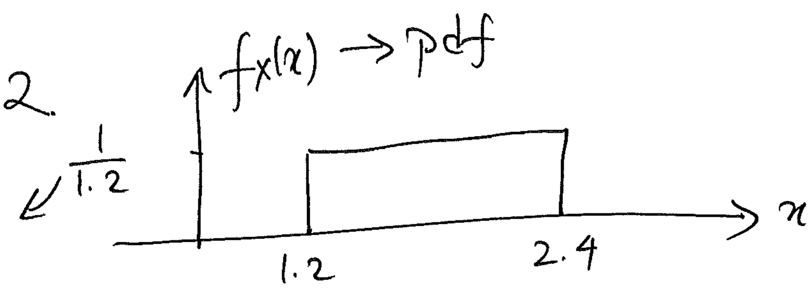
   Hint 2: $\cos(\pi/4) = \frac{\sqrt{2}}{2} \approx 0.71$, $\cos(\pi/3) = 0.5$, and $\cos(\pi/2) = 0$.

   Hint 3: This problem is comparably harder so you might want to come back to this question after you have finished other questions.

1. $x \in [1.2, 2.4]$, $x \in \mathbb{R}$



2.



$\frac{5}{6}$

3. $P\left(\cos\frac{\pi X}{4.8} < 0.5\right)$ $\xrightarrow{} \cos\frac{\pi}{3}$

$= P\left(\frac{\pi X}{4.8} > \frac{\pi}{3}\right)$

$= P(X > 1.6)$

$= \frac{2.4 - 1.6}{2.4 - 1.2} = \frac{0.8}{1.2} = 2/3$ //

Question 7: [11%, Work-out question, Learning Objective 1] Consider a discrete random variable X with the corresponding probability mass function being

$$p_k = \begin{cases} \frac{1}{9} \cdot 0.9^k & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0 \end{cases} \tag{6}$$

Find the probability that $P(X \text{ is a multiple of } 5)$.

P(X is a multiple of 5)

$$= \frac{1}{9}\left((0.9)^5 + (0.9)^{10} \; \ldots \right)$$

$$= \frac{1}{9}\left(\frac{(0.9)^5}{1-(0.9)^5}\right) \; //$$