Final Exam of ECE302, Section 2
1–3pm, Tuesday, May 2, 2017, RHPH172.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Solution

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [12%, Work-out question] Consider two random variables X and Y with joint pdf being

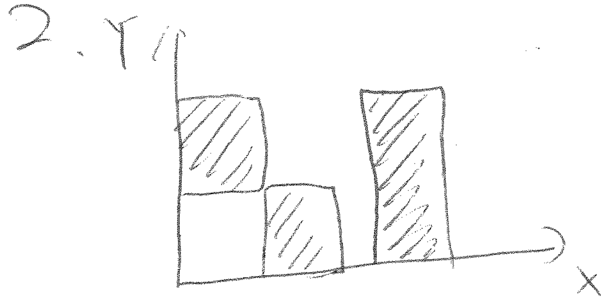
$$f_{X,Y}(x,y) = \begin{cases} 0.25 & \text{if } 0 < x < 1 \text{ and } 1 < y < 2 \\ 0.25 & \text{if } 1 < x < 2 \text{ and } 0 < y < 1 \\ 0.25 & \text{if } 2.5 < x < 3.5 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

1. [1%] What does the acronym "MMSE" stand for?
2. [11%] Find the linear MMSE estimator of X given Y.

   Hint 1: If you do not know how to solve this question, you can find the following values instead: $E(X)$, $E(Y)$, $\mathsf{Var}(X)$, $\mathsf{Var}(Y)$, and $\rho_{X,Y}$ instead. You will receive 8 points if your answers are correct.

1. Minimum Mean Square Error.

2.



$$m_Y = \int_{y=1}^{2}\int_{x=0}^{1} 0.25\,y\,dx\,dy + \int_{y=0}^{1}\int_{x=1}^{2} 0.25\,y\,dx\,dy$$

$$+ \int_{y=0}^{2}\int_{x=2.5}^{3.5} 0.25\,y\,dx\,dy = 1.$$

$$m_X = \int_{y=1}^{2}\int_{x=0}^{1} 0.25\,x\,dx\,dy + \int_{y=0}^{1}\int_{x=1}^{2} 0.25\,x\,dx\,dy$$

$$+ \int_{y=0}^{2}\int_{x=2.5}^{3.5} 0.25\,x\,dx\,dy$$

$$= 2$$

$$\cancel{\sigma_X^2} = E(X^2)$$

$$= \int_{y=1}^{2}\int_{x=0}^{1} 0.25x^2\,dx\,dy$$

$$+ \int_{y=0}^{1}\int_{x=1}^{2} 0.25x^2\,dx\,dy$$

$$+ \int_{y=0}^{2}\int_{x=2.5}^{3.5} 0.25x^2\,dx\,dy$$

$$= 0.25\left(\frac{1}{3}-0\right) + 0.25\left(\frac{8}{3}-\frac{1}{3}\right)$$

$$+ 0.25 \times 2 \times \left(\frac{3.5^3}{3} - \frac{2.5^3}{3}\right)$$

$$= 0.25\cdot\left(\frac{8}{3}\right) + 0.5\cdot\left(\frac{3.5^3}{3} - \frac{2.5^3}{3}\right)$$

$$\cancel{E(Y^2) = \int} \quad \cancel{Var(X) = \frac{2}{3} + \left(\frac{49}{24} - \frac{25}{24}\right)} \quad \cancel{= \frac{2}{3}} \quad -2^2$$

$$= \frac{2}{3} + \frac{7^3 - 5^3}{48} = \frac{32 + 343 - 125}{48} = \frac{250}{48}$$

$$Var(X) = \frac{250}{48} - 4 = \frac{250 - 192}{48} = \frac{58}{48}$$

$$Var(Y) = \frac{(2-0)^2}{12} = \frac{1}{3}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = \int_{y=1}^{2}\int_{x=0}^{1} 0{,}25\,xy\,dx\,dy$$

$$+ \int_{y=0}^{1}\int_{x=1}^{2} 0{,}25\,xy\,dx\,dy$$

$$+ \int_{y=0}^{2}\int_{x=2{,}5}^{3{,}5} 0{,}25\,xy\,dx\,dy$$

$$= 0{,}25 \cdot \frac{1}{2} \cdot \frac{3}{2}$$

$$+ 0{,}25 \times \frac{1}{2} \times \frac{3}{2}$$

$$+ 0{,}25 \times \frac{4}{2} \cdot \frac{3{,}5^2 - 2{,}5^2}{2}$$

$$= \frac{3}{8} + \frac{1}{2} \cdot \frac{49-25}{8} = \frac{3}{8} + \frac{3}{2} = \frac{15}{8}$$

$$\mathrm{Cov}(X,Y) = \frac{15}{8} - 2 \times 1 = -\frac{1}{8}$$

$$\rho = \frac{-\frac{1}{8}}{\sqrt{\frac{1}{3} \cdot \frac{58}{48}}}$$

$$\hat{X}_{LINMMSE} = \frac{-\frac{1}{8}}{\sqrt{\frac{1}{3} \cdot \frac{58}{48}}} \cdot \frac{\sqrt{\frac{58}{48}}}{\sqrt{\frac{1}{3}}} \cdot (y-1) + 2$$

Question 2: [13%, Work-out question] X_1, X_2, X_3 are independent standard Gaussian random variables. Using these three random variables, we can construct another three random variables by

$$U = X_1 + 3X_2 - 2$$
$$V = 2X_1 - 2X_2$$
$$W = X_3 + 1$$

1. [2%] Are W and U independent? Are W and V independent?
2. [4%] Express the probability $P(U < 3)$ in terms of the Q function. That is $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.
3. [7%] Write down the joint pdf of (U, V, W). Your answer can be expressed as some matrix A. However, you cannot leave an inverse matrix (e.g., A^{-1}) in your expression. If you see A^{-1}, you need to find out exactly what is the expression of A^{-1} and cannot leave it as A^{-1}.

   Hint 1: If you do not know how to compute the inverse, you can leave A^{-1} in your expression. However, you will receive 5 points if all the rest of the answer is correct.

   Hint 2: If you do not know the answer, you can compute the mean vector and covariance vectors of (U, V, W) instead. You will receive 4 points if your answer is correct.

   Hint 3: It may be easier if you first find out the joint pdf of (U, V) and then use it to derive the joint pdf of (U, V, W).

1. Yes. Yes.

2. $m_U = -2$. $\sigma_U^2 = 1^2 + 3^2 = 10$,

$$P(U<3) = P(Z \cdot \sqrt{10} - 2 < 3)$$
$$= P\left(Z < \frac{5}{\sqrt{10}}\right)$$
$$= 1 - Q\left(\frac{5}{\sqrt{10}}\right)$$

3. $m_V = 0$ $\quad \sigma_V^2 = 4+4=8.$

$\mathrm{Cov}(U,V) = 2-6 = -4.$

$$\rho = \frac{-4}{\sqrt{10\times 8}} = \frac{-4}{\sqrt{80}}.$$

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}^2 \cdot \sqrt{8}\times\sqrt{10}\cdot\sqrt{\left(1-\left(\frac{16}{80}\right)^2\right)}} \cdot e^{-\frac{\left(\frac{u+2}{\sqrt{10}}\right)^2 - 2\cdot\left(\frac{-4}{\sqrt{80}}\right)\cdot\left(\frac{u+2}{\sqrt{10}}\right)\left(\frac{v}{\sqrt{8}}\right) + \left(\frac{v}{\sqrt{8}}\right)^2}{2\cdot\left(1-\left(\frac{16}{80}\right)^2\right)}}$$

$f_{U,V,W}(u,v,w)$

$$= \frac{1}{\sqrt{2\pi}^2\times\sqrt{8}\times\sqrt{10}\cdot\sqrt{1-\left(\frac{4}{80}\right)^2}} e^{-\frac{\left(\frac{u+2}{\sqrt{10}}\right)^2 - 2\cdot\left(\frac{-4}{80}\right)\left(\frac{u+2}{\sqrt{10}}\right)\left(\frac{v}{\sqrt{8}}\right) + \left(\frac{v}{\sqrt{8}}\right)^2}{2\left(1-\left(\frac{4}{80}\right)^2\right)}}$$

$$\cdot \frac{1}{\sqrt{2\pi\times 1}} e^{-\frac{(w-1)^2}{2\cdot 1}}$$

#

Question 3: [14%, Work-out question] Denote the number of webpage requests currently being processed by a given server as X. For example $X = 3$ means that the server is current processing 3 webpage requests simultaneously. We use Y to denote the ping response time of the same server. Namely, $Y = 3.25$ms means that after we send a ping request to the server, we will receive the response in exactly 3.25ms.

We assume X is a binomial random variable with parameter $n = 3$ and $p = \frac{3}{4}$. We further assume that if the server is processing $X = x_0$ requests simultaneously, the ping response time Y is uniformly distributed between 0ms to $x_0 \times 2.5$ms. For example, if the server is currently processing $X = 3$ requests simultaneously, then the ping response time will be uniformly distributed between 0ms to $3 \times 2.5\text{ms} = 7.5$ms. If the server is currently processing no request ($X = 0$), then the ping response time will be very very short and can thus be represented by 0ms.

One way to justify this model is that the more webpage requests being processed, the longer the average response time would be.

1. [6%] Suppose we send a ping and the response time is $Y = 2$ms. What is the ML detector of X given $Y = 2$ms?

   Hint: If you do not know how to solve this problem, you should explain what the acronym ML stands for and how you plan to approach this problem. You will receive 4 points if your answers are correct.

2. [8%] Suppose we send a ping and the response time is $Y = 2$ms. What is the MAP detector of X given $Y = 2$ms?

   Hint: If you do not know how to solve this problem, you should explain what the acronym MAP stands for and how you plan to approach this problem. You will receive 5 points if your answers are correct.

1. $P(Y=2 \mid X=0) = 0$

$P(Y=2 \mid X=1) = \frac{1}{2.5 \times 1}\,dy$

$P(Y=2 \mid X=2) = \frac{1}{2.5 \times 2}\,dy$

$P(Y=2 \mid X=3) = \frac{1}{2.5 \times 3}\,dy$

$\Rightarrow \hat{X}_{ML}(Y=2) = 1$

$2 \sim P(X=0 \mid Y=2) = 0$

$P(X=1 \mid Y=2) =$

$$= \frac{\frac{1}{2.5}\binom{3}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^2}{\frac{1}{2.5}\times\binom{3}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^2 + \frac{1}{2.5\times2}\binom{3}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^1 + \frac{1}{2.5\times3}\binom{3}{3}\left(\frac{3}{4}\right)^3}$$

$$= \frac{6\times3\times3}{6\times3\times3 + 3\times3\times3^2 + 2\times3^3}$$

$$= \frac{2}{2+3+2} = \frac{2}{7}$$

$P(X=2 \mid Y=2) = \frac{3}{2+3+2} = \frac{3}{7}$,

$P(X=3 \mid Y=2) = \frac{2}{2+3+2} = \frac{2}{7}$

$X_{MAP}(Y=2) = 2.$

Question 4: [12%, Work-out question] Suppose X_1, X_2, ..., X_n, ... are independently and identically distributed binomial random variables with $(n,p) = (4, 0.1)$. Define the *sample mean* $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. For example $M_{10000} = \frac{1}{10000}\sum_{i=1}^{10000} X_i$.

1. [4%] Express the probability $P(M_{10000} \geq 0.412)$ as a summation. Hint: You do not need to expand/compute the value of the summation. Something like $\sum_k \frac{1}{3^k}$ would suffice.

   Hint: You may want to find out the distribution of the intermediate random variable $S_{10000} = \sum_{i=1}^{10000} X_i$.

2. [8%] Use the central limit theorem and Gaussian approximation to express the probability $P(M_{10000} \geq 0.412)$ using the Q function. That is $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.

   Hint 1: The answer of this sub-question does not depend on the answer to the previous sub-question.

   Hint 2: If you do not know how to solve this problem, you should clearly explain what is the central limit theorem. You will receive 5 points if your answer is correct.

1. $P(M_{10000} \geq 0.412)$

$= P(S_{10000} \geq 4120)$

$= \sum_{k=4120}^{40000} \binom{40000}{k} (0.1)^k \cdot (0.9)^{40000-k}$

2. S_{10000} has mean $10000 \times (4 \times 0.1)$

$= \cancel{40000}\ 4000$

Variance $= 10000 \times 4 \times 0.1 \times 0.9$

$= 3600$

$Z_{10000} = \frac{S_{10000} - 4000}{\sqrt{3600}}$

$P(S_{10000} \geq 4120) \approx P(Z_{10000} \geq \frac{120}{60}) \approx Q(2)$

Question 5: [13%, Work-out question] Suppose X_1, X_2, ..., X_n, ... are independently and identically distributed Benoulli random variables with $p = 0.5$.

We construct a random process Y_n such that $Y_n = X_{n-1} + X_n$. For example $Y_2 = X_1 + X_2$ and $Y_{29} = X_{28} + X_{29}$ and so on so forth.

1. [4%] Find out the probability that $P(Y_1 \geq 1 \text{ and } Y_2 \geq 1)$.

   Hint: $Y_1 = X_0 + X_1$ and $Y_2 = X_1 + X_2$.

2. [2%] Find out the mean function of Y_n. That is, what is the function $m_Y(n) = E(Y_n)$.

   Hint: Do not be scared by the use of n. You should simply consider $n = 1$ first and try to find out $m_Y(1)$. You can then generalize to the case of $m_Y(n)$ for arbitrary n.

3. [1%] Write down the definition of the auto-covariance function $C_Y(n_1, n_2)$ of Y.

4. [3%] Compute the auto-covariance function value $C_Y(100, 101)$.

   Hint 1: $Y_{100} = X_{99} + X_{100}$ and $Y_{101} = X_{100} + X_{101}$.

   Hint 2: If you do not know what is an auto-covariance function, you can simply compute the value of $E(Y_{100}Y_{101})$ you will receive 2 points if your answer is correct.

5. [3%] Compute the auto-covariance function values of $C_Y(100, 102)$ and $C_Y(100, 103)$.

   Hint 2: If you do not know what is an auto-covariance function, you can simply compute the value of $E(Y_{10}Y_{11}Y_{12})$ you will receive 3 points if your answer is correct.

1.

X_0 \ X_1X_2	00	01	10	11
0			✓	✓
1		✓	✓	✓

$P(Y_1 \geq 1 \ \& \ Y_2 \geq 1) = 5/8.$

2. $E(Y_n) = E(X_{n-1}) + E(X_n) = 0.5 + 0.5 = 1$

$$\boxed{\begin{aligned} &E(Y_{10}Y_{11}Y_{12}) = 1.5 \\ &E(Y_{100}Y_{101}) = 5/4 \end{aligned}}$$

3. $C_Y(n_1, n_2) = Cov(Y_{n_1}, Y_{n_2})$

4. $C_Y(100, 101) = Cov(Y_{100}, Y_{101}) = Var(X_1) = 0.25$

5. $C_Y(100, 102) = 0 = C_Y(100, 103)$

Question 6: [10%, Work-out question] Suppose X_1 is an exponential random variable with $\lambda = 1$ and X_2 is an exponential random variable with $\lambda = 2$. Also suppose X_1 and X_2 are independent. We construct a new random variable $Y = \max(X_1, X_2)$.

Question: Find out the pdf of Y.

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(\max(X_1, X_2) \le y) \\
&= P(X_1 \le y \ \& \ X_2 \le y) \\
&= P(X_1 \le y) \cdot P(X_2 \le y) \\
&= (1 - e^{-y}) \cdot (1 - e^{-2y}) \\
&= 1 - e^{-y} - e^{-2y} + e^{-3y} \quad \text{if } 0 \le y.
\end{aligned}
$$

$$
f_Y(y) = \begin{cases} e^{y} + 2e^{-2y} - 3e^{-3y} & \text{if } 0 \le y \\ 0 & \text{otherwise.} \end{cases}
$$

Question 7: [8%, Work-out question] Suppose a jewelry store, in average, has 0.75 customer arrivals per hour. Let X denote the number of customers arriving at this jeweler between 9am and 5pm. Also assume that if there are $X = x_0$ number of customers during 9am to 5pm, the total sales of this $2^{x_0} \times \$1000$. For example, if there are $X = 3$ customers, then the total sales for that day is $2^3 \times \$1000 = \$8000$.

Question: What is the probability that there is no customer arriving today, given that the total sales number is no larger than \$9999 today.

$$\alpha = 0.75 \times 8 = 6.$$

$$P(X=0 \mid 2^X \times 1000 \le 4999)$$

$$= P(X=0 \mid X \le \cancel{8}\, 2)$$

$$= \frac{\frac{6^0}{0!} e^{-6}}{\frac{6^0}{0!} e^{-6} + \frac{6^1}{1!} e^{-6} + \frac{6^2}{2!} e^{-6} + \cancel{\frac{6^3}{3!} e^{-6}}}$$

$$= \frac{1}{1 + 6 + 18}$$

$$= \cancel{\frac{1}{61}} = \frac{1}{25}$$

Question 8: [18%,True/false question. There is no need to justify your answers]
Decide whether the following statements are true or false.

No

1. [2%] X and Y are uniformly distributed in a unit circle centered at the $(-2,-3)$, i.e., those (x,y) satisfying $(x+2)^2+(y+3)^2 \leq 1$. The random variables X and Y are orthogonal.

Yes.

2. [2%] We use X_1 to X_{100} to denote 100 different random variables. Each X_i has mean zero $E(X_i)=0$ unit variance $\mathsf{Var}(X)=1$ and any two of them has covariance $\mathsf{Cov}(X_i,X_j)=1$. Let $Y=\frac{1}{100}\sum_{i=1}^{100}(X_i)^2$. We must have $E(Y)=1$.

No,

3. [2%] Both X and Y are standard Gaussian random variable and X and Y are orthognal. Therefore X and Y must also be independent.

No

4. [2%] Suppose X_1 is a Bernoulli random variable with $p=0.5$ and (X_1,X_2) have correlation coefficient being exactly 1. X_2 must also be a Bernoulli random variable with $p = \cancel{1}$. 0.5

Yes.

5. [2%] If X and Y are uncorrelated, then we always have $E(XY)=E(X)E(Y)$.

Yes.

6. [2%] Consider $F_Y(y)$ to be the cdf of random variable Y. We must have $0 \leq F_Y(y) \leq 1$.

No.

7. [2%] X is a Gaussian random variable with $m=3$ and $\sigma^2=40000$. By the Markov inequality, we must have $P(X>10) \leq \frac{E(X)}{10}=0.3$.

No.

8. [2%] X_1 is a Poisson random variable with $\alpha_1=3$ and X_2 is a Poisson random variable with $\alpha_2=2$ and the correlation coefficient between X_1 and X_2 is 0.5. The new variable $Y=X_1+X_2$ is a Poisson random variable with $\alpha_Y=0.5*(\alpha_1+\alpha_2)$.

No.

9. [2%] Consider two random variables X and Y with the corresponding cdf $F_X(x)$ and $F_Y(y)$. Suppose we know that $P(X>Y)=1$. We must have $F_X(1)<F_Y(1)$.