Final Exam of ECE302, Section 2

1–3pm, Tuesday, May 2, 2017, RHPH172.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [12%, Work-out question] Consider two random variables X and Y with joint pdf being

$$f_{X,Y}(x,y) = \begin{cases} 0.25 & \text{if } 0 < x < 1 \text{ and } 1 < y < 2 \\ 0.25 & \text{if } 1 < x < 2 \text{ and } 0 < y < 1 \\ 0.25 & \text{if } 2.5 < x < 3.5 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

We have already known that $m_X = 2$ and $\mathsf{Var}(X) = \frac{29}{24}$. Answer the following questions.

1. [1%] What does the acronym "MMSE" stand for?

2. [11%] Find the linear MMSE estimator of X given Y.

   Hint 1: If you do not know how to solve this question, you can find the following values instead: $E(X)$, $E(Y)$, $\mathsf{Var}(X)$, $\mathsf{Var}(Y)$, and $\rho_{X,Y}$ instead. You will receive 8 points if your answers are correct.

Question 2: [13%, Work-out question] X_1 and X_2 are independent standard Gaussian random variables. Using these two random variables, we can construct another two random variables by

$$
\begin{aligned}
U &= X_1 + 3X_2 - 2 \\
V &= 2X_1 - 2X_2
\end{aligned}
$$

1. [5%] Express the probability $P(U < 3)$ in terms of the Q function. That is $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.
2. [8%] Write down the joint pdf of (U, V).

   Hint 1: The joint pdf of 2-dimensional Gaussian random variables is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)}} e^{-\frac{\left(\frac{x-m_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right) + \left(\frac{y-m_Y}{\sigma_Y}\right)^2}{2(1-\rho^2)}} \tag{2}$$

Question 3: [14%, Work-out question] Denote the number of webpage requests currently being processed by a given server as X. For example $X = 3$ means that the server is current processing 3 webpage requests simultaneously. We use Y to denote the ping response time of the same server. Namely, $Y = 3.25$ms means that after we send a ping request to the server, we will receive the response in exactly 3.25ms.

We assume X is a binomial random variable with parameter $n = 3$ and $p = \frac{3}{4}$. We further assume that if the server is processing $X = x_0$ requests simultaneously, the ping response time Y is uniformly distributed between 0ms to $x_0 \times 2.5$ms. For example, if the server is currently processing $X = 3$ requests simultaneously, then the ping response time will be uniformly distributed between 0ms to $3 \times 2.5\text{ms} = 7.5$ms. If the server is currently processing no request ($X = 0$), then the ping response time will be very very short and can thus be represented by 0ms.

One way to justify this model is that the more webpage requests being processed, the longer the average response time would be.

1. [6%] Suppose we send a ping and the response time is $Y = 2$ms. What is the ML detector of X given $Y = 2$ms?

   Hint: If you do not know how to solve this problem, you should explain what the acronym ML stands for and how you plan to approach this problem. You will receive 4 points if your answers are correct.

2. [8%] Suppose we send a ping and the response time is $Y = 2$ms. What is the MAP detector of X given $Y = 2$ms?

   Hint: If you do not know how to solve this problem, you should explain what the acronym MAP stands for and how you plan to approach this problem. You will receive 5 points if your answers are correct.

Question 4: [9%, Work-out question] Suppose X_1, X_2, ..., X_n, ... are independently and identically distributed binomial random variables with $(n, p) = (4, 0.1)$. Define the *sample mean* $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. For example $M_{10000} = \frac{1}{10000}\sum_{i=1}^{10000} X_i$.

1. [9%] Use the central limit theorem and Gaussian approximation to express the probability $P(M_{10000} \geq 0.412)$ using the Q function. That is $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$.

   Hint 1: You may want to start from the intermediate random variable $S_{10000} = \sum_{i=1}^{10000} X_i$ instead of M_{10000}.

   Hint 2: If you do not know how to solve this problem, you should clearly explain what is the central limit theorem. You will receive 6 points if your answer is correct.

Question 5: [14%, Work-out question] Suppose X_1, X_2, ..., X_n, ... are independently and identically distributed Benoulli random variables with $p = 0.5$.

We construct a random process Y_n such that $Y_n = X_{n-1} + X_n$. For example $Y_2 = X_1 + X_2$ and $Y_{29} = X_{28} + X_{29}$ and so on so forth.

1. [5%] Find out the probability that $P(Y_1 \geq 1 \text{ and } Y_2 \geq 1)$.

   Hint: $Y_1 = X_0 + X_1$ and $Y_2 = X_1 + X_2$.

2. [1%] Write down the definition of the auto-covariance function $C_Y(n_1, n_2)$ of Y.

3. [5%] Compute the auto-covariance function value $C_Y(100, 101)$.

   Hint 1: $Y_{100} = X_{99} + X_{100}$ and $Y_{101} = X_{100} + X_{101}$.

   Hint 2: If you do not know what is an auto-covariance function, you can simply compute the value of $E(Y_{100}Y_{101})$ you will receive 3 points if your answer is correct.

4. [3%] Compute the auto-covariance function value of $C_Y(100, 102)$.

   Hint 3: If you do not know what is an auto-covariance function, you can simply compute the value of $E(Y_{10}Y_{11}Y_{12})$ you will receive 3 points if your answer is correct.

Question 6: [10%, Work-out question] Suppose X_1 is an exponential random variable with $\lambda = 1$ and X_2 is an exponential random variable with $\lambda = 2$. Also suppose X_1 and X_2 are independent. We construct a new random variable $Y = \max(X_1, X_2)$.

Question: Find out the pdf of Y.

Question 7: [10%, Work-out question] Suppose a jewelry store, in average, has 0.75 customer arrivals per hour. Let X denote the number of customers arriving at this jeweler between 9am and 5pm. Also assume that if there are $X = x_0$ number of customers during 9am to 5pm, the total sales of this $2^{x_0} \times \$1000$. For example, if there are $X = 3$ customers, then the total sales for that day is $2^3 \times \$1000 = \$8000$.

Question: What is the probability that there is no customer arriving today, given that the total sales number is no larger than \$4999 today. Your answer must be a single number. You cannot leave it as a fraction in this question. For example, you should not leave your answer to be something like $\frac{1}{50}$. Instead, you should write it as 0.02 instead.

Hint: If your answer is a fraction then you will receive 7 points instead.

Question 8: [18%,True/false question. There is no need to justify your answers]
Decide whether the following statements are true or false.

1. [2%] X and Y are uniformly distributed in a unit circle centered at the $(-2,-3)$, i.e., those (x,y) satisfying $(x+2)^2+(y+3)^2 \leq 1$. The random variables X and Y are orthogonal.

2. [2%] We use X_1 to X_{100} to denote 100 different random variables. Each X_i has mean zero $E(X_i)=0$ unit variance $\mathsf{Var}(X)=1$ and any two of them has covariance $\mathsf{Cov}(X_i,X_j)=1$. Let $Y=\frac{1}{100}\sum_{i=1}^{100}(X_i)^2$. We must have $E(Y)=1$.

3. [2%] Both X and Y are standard Gaussian random variable and X and Y are orthognal. Therefore X and Y must also be independent.

4. [2%] Suppose X_1 is a Bernoulli random variable with $p=0.5$ and (X_1,X_2) have correlation coefficient being exactly 1. X_2 must also be a Bernoulli random variable with $p=0.5$.

5. [2%] If X and Y are uncorrelated, then we always have $E(XY)=E(X)E(Y)$.

6. [2%] Consider $F_Y(y)$ to be the cdf of random variable Y. We must have $0 \leq F_Y(y) \leq 1$.

7. [2%] X is a Gaussian random variable with $m=3$ and $\sigma^2=40000$. By the Markov inequality, we must have $P(X \geq 10) \leq \frac{E(X)}{10} = 0.3$.

8. [2%] X_1 is a Poisson random variable with $\alpha_1=3$ and X_2 is a Poisson random variable with $\alpha_2=2$ and the correlation coefficient between X_1 and X_2 is 0.5. The new variable $Y=X_1+X_2$ is a Poisson random variable with $\alpha_Y=0.5*(\alpha_1+\alpha_2)$.

9. [2%] Consider two random variables X and Y with the corresponding cdf $F_X(x)$ and $F_Y(y)$. Suppose we know that $P(X>Y)=1$. We must have $F_X(1)<F_Y(1)$.

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\text{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\text{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\text{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\text{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega} - 1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \leq x \leq b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.

- 2-dimensional Gaussian Random Vector

  $S = \{(x, y) : \text{for all real-valued } x \text{ and } y\}$

  $f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)}$

  $E(X) = m_X$, $\mathrm{Var}(X) = \sigma_X^2$, $E(Y) = m_Y$, $\mathrm{Var}(Y) = \sigma_Y^2$, and $\mathrm{Cov}(X, Y) = \rho\sqrt{\sigma_X^2\sigma_Y^2}$.

- n-dimensional Gaussian Random Variable

  $S = \{(x_1, x_2, \cdots, x_n) : \text{for all real-valued } x_1 \text{ to } x_n\}$

  If we denote $\vec{x} = (x_1, x_2, \cdots, x_n)$ as an n-dimensional row-vector, then the pdf of an n-dimensional Gaussian random vector becomes

  $f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(K)}} e^{-\frac{1}{2}(\vec{x}-\vec{m})K^{-1}(\vec{x}-\vec{m})^{\mathrm{T}}}$

  where $\vec{m}$ is the mean vector of X, i.e., $\vec{m} = E(\vec{X})$; K is an $n \times n$ covariance matrix, where the (i, j)-th entry of the K matrix is $\mathrm{Cov}(X_i, X_j)$; $\det(K)$ is the determinant of K; and K^{-1} is the inverse of K.