Midterm #3 of ECE302, Section 3

6:30–7:30pm, Wednesday, April 06, 2016, EE 117.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [25%, Work-out question] Suppose random variable X is Poisson with parameter $\alpha = 1$ and random variable Y is Poisson with parameter $\alpha = 2$. Also suppose that X and Y are independent.

1. [8%] Find the joint sample space $S_{X,Y}$ and the corresponding joint pmf $p_{x,y}$.
2. [8%] Find the probability that $P(X^2 2^Y < 3)$
3. [9%] Find the conditional expectation $E(X|X+Y=1)$.
4. [Bonus question 5%] Find the conditional expectation $E(X|X+Y=n)$ for arbitrary n.

   Hint: This question is considerably harder. You should work on this bonus question only if you have answered all other questions of this exam correctly.

Question 2: [20%, Work-out question]

Consider a continuous random variable X with pdf being

$$f_X(x) = 3e^{-6|x|} \quad \text{for all } -\infty < x < \infty \tag{1}$$

1. [10%] Find the characteristics function $\Phi_X(\omega)$ of X.

   Hint: If you do not know how to find a characteristic function, you can find the following expectation $E(e^{-sX})$ instead. You will receive 8 points if your answer is correct.

2. [10%] Use the characteristics function $\Phi_X(\omega)$ to find the variance of X.

   Hint 1: If you do not know the answer to the previous sub-question, you can assume that $\Phi_X(\omega) = \frac{4}{4+\omega^2}$. You will still get 10 points if your answer is correct.

   Hint 2: If you do not know how to answer this question, you can use any other method to solve the variance of X. You will receive 6 points if your answer is correct.

Question 3: [20%, Work-out question]

X is Gaussian random variable with $m = 5$ and $\sigma^2 = 16$. $Y = 2X - 9$.

1. [5%] Find the mean and variance of Y.

2. [15%] Find the probability $P(|\log_2(Y)| < 1)$ in terms of the Q function. That is, $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$.

   Hint 1: $\log_2$ is the logarithmic function with base 2. For example, $\log_2(1) = 0$, $\log_2(2) = 1$, $\log_2(16) = 4$, and $\log_2(\frac{1}{8}) = -3$.

   Hint 2: If you do not know how to solve this question, you can find the probability $P(|X| < 3)$ in terms of the Q function. You will receive 10 points if your answer is correct.

Question 4: [15%, Work-out question] The continuous random variable Y is uniformly distributed on $(1, 4)$; a continuous random variable X is uniformly distributed on $(0, 2)$; and X and Y are independent.

[15%] Find the joint cdf $F_{X,Y}(x, y)$.

Hint: It might be easier to directly compute the "volume = base × height" rather than carrying out the 2-D integral.

Question 5: [20%, Multiple choice question. There is no need to justify your answers]

1. [2%] X and Y are uniformly distributed in a square centered at the origin, i.e., those (x, y) satisfying $|x| < 1$ and $|y| < 1$. Compute $R = \sqrt{X^2 + Y^2}$ as the radius and Θ is the corresponding angle (ranging from 0 to 2π). Are R and Θ independent?

2. [2%] R is uniformly distributed on $(0, 1)$; Θ is Bernoulli distributed with parameter $p = 0.5$; and R and Θ are independent. Compute $X = R\cos(\pi\Theta)$ and $Y = R\sin(\pi\Theta)$. Are X and Y independent?

3. [2%] R is uniformly distributed on $(0, 1)$; Θ is Bernoulli distributed with parameter $p = 0.5$; and R and Θ are independent. Compute $X = R\cos(0.5\pi\Theta)$ and $Y = R\sin(0.5\pi\Theta)$. Are X and Y independent?

4. [2%] Consider a random variable X with mean $m = 3$ and variance $\sigma^2 = 36$. Is the following inequality always true: $P(X \geq 10) \leq \frac{m}{10}$?

5. [2%] Consider a random variable X with mean $m = 3$ and variance $\sigma^2 = 36$. Is the following inequality always true: $P((X - m) \geq 10) \leq \frac{\sigma^2}{100}$?

6. [2%] Consider a random variable X with moment generating function $X^*(s) = E(e^{-sX})$. Is the following inequality always true: $P(X \geq 10) \leq e^{-2\cdot 10} X^*(-2)$?

7. [3%] Suppose X is Poisson with parameter $\alpha = 2$ and Y is binomial with parameter $n = 5$ and $p = 0.7$. Also suppose that X and Y are independent. Consider the corresponding joint cdf $F_{X,Y}(x, y)$. Does the inequality $F_{X,Y}(101, 10) < F_{X,Y}(101, 21)$ hold?

8. [3%] Is the following statement always true? "If two random variables X and Y are independent, then we have $E(X^2Y^3) = (E(X))^2(E(Y))^3$."

9. [2%] Is the following statement always true? "If both X and Y are exponentially distributed with the same parameter value $\lambda_X = \lambda_Y = 3$, then X and Y are independent."

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\mathrm{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\mathrm{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega}-1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \le x \le b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.