Midterm #2 of ECE302, Section 3

6:30–7:30pm, Wednesday, March 02, 2016, EE 117.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [20%, Work-out question, Learning Objective 1]

1. [3%] X is a uniform random variable with $(a, b) = (3, \pi)$. Find the expectation $E(X)$.

2. [6%] X is a geometric random variable with $p = \frac{1}{3}$. Find the conditional probability $P(X \leq 10 | X \geq 3)$.

3. [8%] X is an exponential random variable with $\lambda = 3$. Suppose $Y = 3X + 4$ find the values of $E(Y)$, $E(Y^2)$, and $\mathsf{Var}(Y)$.

4. [3%] X is a binomial random variable with $(n, p) = (10, \frac{2}{3})$. Find the probability $P(X$ is a prime number$)$.
   Hint 1: You do not need to expand the expression. For example, your answer can be something like $0.5^{10} \cdot \frac{6!}{4!} + 0.5$. There is no need to compute the final value.

   Hint 2: the smallest prime number is 2.

Question 2: [23%, Work-out question, Learning Objective 1]

1. [2%] What does the acronym "cdf" stand for?

Consider a 2-faced fair die. Namely, the face value can be 1 or 2, with equal probability. Suppose we first throw the 2-faced die and denote the random outcome (the face value) as X. Then we run a computer algorithm that generates a random number Y *uniformly randomly* between 0 and $\pi \cdot X$. (Or equivalently, Y will be uniformly distributed between 0 and $\pi \cdot X$.) For example, if the initial outcome is $X = 2$, then the computer will uniformly generate a random number between 0 and 2π.

2. [15%] Find the cdf $F_Y(y)$ of Y and plot it for the range of $-\pi < y < 5\pi$.

   Hint: The following equation $P(a \leq Y \leq b) = P(a \leq Y \leq b, X = 1) + P(a \leq Y \leq b, X = 2)$ may be useful.

3. [1.5%] Is the random variable Y continuous or discrete?

4. [4.5%] Find the pdf $f_Y(y)$ of Y and plot it for the range of $-\pi < y < 5\pi$.

   Hint: If you do not know the answer to Q2.2, you can assume

$$F_Y(y) = \begin{cases} 0 & \text{if } y < -0.5\pi \\ 0.5 + 0.25\sin(y) & \text{if } -0.5\pi \leq y < 0.5\pi \\ 1 & \text{if } 0.5\pi \leq y \end{cases} \tag{1}$$

   and use it to answer both Q2.3 and Q2.4. You will get full credit (1.5 pts and 4.5 pts) if your answers are correct.

Question 3: [25%, Work-out question, Learning Objective 1] Consider a continuous random variable X that is uniformly distributed between 0 and 1. Consider the following function

$$f(x) = \begin{cases} -1 & \text{if } 0 \le x < 0.5 \\ 4x - 3 & \text{if } 0.5 \le x < 1 \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

1. [2%] Plot $f(x)$ versus x for the range of $-2 < x < 2$.
2. [15%] Using the above function, consider another random variable $Y = f(X)$. Find the pdf $f_Y(y)$ of Y and plot it for the range of $-2 < y < 2$.

   Hint: You may want to find the cdf of Y first.
3. [8%] Find the expectation $E(Y)$.

   Hint: If you do not know the answer to the previous subquestion, you can assume

   $$f_Y(y) = \sum_{k=2}^{3} 2^{-k}\delta(y-k) + \begin{cases} \frac{5}{32} & \text{if } -2 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

   You will still get full credit (8 pts) if your answer is correct.

Question 4: [18%, Work-out question, Learning Objective 1] We analyze the so-called distributed-denial-of-service (DDoS) attack.

Suppose that the web-page server of Purdue will crash if there are ≥ 100 requests arriving at the server in 10 second. Also suppose that IF a hacker has gained control of 1 PC, then that PC will send "web-page request" to Purdue server at a rate of 30 request per minutes. In this question, we assume the web-page request is Poisson distributed.

1. [9%] Suppose that the hacker controls only 1 PC, what is the probability that the web server crashes after 10 second?

   Hint: You do not need to compute the final number. A carefully written sum/interation formula that explains your idea should suffice.

2. [5%] Suppose that the hacker controls only 10 PCs, what is the probability that the web server crashes after 10 second?

   Hint: You do not need to compute the final number. A carefully written sum/interation formula that explains your idea should suffice.

3. [4%] If you are the hacker, how many computers, denoted by n, you would need to control so that it is guaranteed that with probability 99% that the server will crash after 10 seconds. Explain that how you plan to find the value of n.

   Hint: A carefully outlined procedure how you plan to find the value of n would suffice.

Question 5: [14%, Work-out question, Learning Objective 1]

X is an exponential random variable with parameter $\lambda = 2$. Let Y denote the integer part of X. For example if $X = 0.112$ then $Y = 0$. If $X = \pi$, then $Y = 3$.

Find the expectation $E(Y)$.

Hint 1: You may want to find the pmf of Y first.

Hint 2: The formula/RV tables may be useful.

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \leq p \leq 1$.

  $E(X) = p$, $\mathrm{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\mathrm{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega}-1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \leq x \leq b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.