Midterm #1 of ECE302, Section 3

6:30–7:30pm, Wednesday, February 03, 2016, ME 1009.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [16%, Work-out question, Learning Objective 1] Consider a continuous function $f_X(x)$.

$$f_X(x) = \begin{cases} 2e^{-x} & \text{if } \ln(2) \leq x \\ 0 & \text{otherwise} \end{cases}$$

Note that $\ln(2) \approx 0.693$. Define the expression of

$$M(s) = \int_{x=-\infty}^{\infty} e^{sx} f_X(x) dx.$$

1. [7%] Find the expression of $M(s)$.
2. [9%] Find the value of $M'(0)$ where $M'(s) = \frac{d}{ds}M(s)$ is the first-order derivative of $M(s)$.

Question 2: [12%, Work-out question, Learning Objective 1] Consider a discrete function $f(k)$ (k being integer) as follows.

$$f(k) = \begin{cases} 0.8 \cdot 0.2^{(k-10)} & \text{if } 10 \leq k \\ 0 & \text{otherwise} \end{cases}$$

Find the value of

$$\sum_{k=-\infty}^{\infty} (2k+9) f(k). \tag{1}$$

Question 3: [15%, Work-out question, Learning Objective 1] Consider a 2-D function.

$$f(x,y) = \begin{cases} x \cdot e^{-x|y|} & \text{if } 0 \leq x < 1 \text{ and } 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

We generate another function

$$F(x,y) = \int_{s=-\infty}^{x} \int_{t=-\infty}^{y} f(s,t) dt ds. \tag{3}$$

1. [12%] Find the value of $F(1.5, 1)$.

2. [3%] Which one is larger, $F(0.5, 1)$ or $F(1.5, 1)$? Please use 1 sentence to justify your answer.

   Hint: You should be able to answer this question without knowing the value of $F(0.5, 1)$.

Question 4: [12%, Work-out question, Learning Objective 1] Consider a three-faced die with each face value being 1, 2, and 3, respectively. We throw the this three-faced die and let X denote the outcome of the face value.

Suppose we know that $P(X > 1.5)$ is twice as large as $P(X \leq 2.1)$. We also know that $P(X = 1) = \frac{1}{7}$.

What is the probability $P(X = 3)$?

If you do not know the answer to this question, you can answer "what are the three axioms of probability". If your answer is correct, you will receive 6 points.

Question 5: [15%, Work-out question, Learning Objective 1]

We flip a fair 3-faced die and use X to denote the face value of the outcome, which can be 1 or 2 or 3.

After knowing the face value of X, then we bend a coin to certain degree so that the probability of the coin satisfies

$$P(Y=1) = \frac{X}{4}$$
$$P(Y=0) = \frac{4-X}{4}$$

where $Y=1$ means that the outcome of the coin is "head" and $Y=0$ means that the outcome of the coin is "tail". For example, if the face value of the die is $X=1$, then we bend the coin so that the head probability is $\frac{1}{4}=0.25$ and the tail probability is $\frac{4-1}{4}=0.75$.

1. [3%] What is the sample space in this experiment?
2. [6%] Use either the tree method or the table method to assign the probabilistic weight.
3. [6%] What is the probability $P(\max(X, 4Y) > 2.1)$?

   Hint: If you do not know the answer to the previous two subquestions, you can assume that the coin is always fair (we do not bend the coin in anyway). You will get 5 points if your answer is correct.

Question 6: [18%, Work-out question, Learning Objective 1] Consider a continuous random variable X, which is uniformly randomly chosen from the interval $[1, 3.5]$. (Being continuous means that X can take any values in the range, for example X can be $\sqrt{2}$.)

1. [2%] What is the sample space?
2. [4%] What is the probability density function (pdf) you will use to describe the probabilistic weight assignment for X?
3. [3%] Find the probability that $P(X < \sqrt{2})$. Hint: $\sqrt{2} \approx 1.414$.
4. [4%] Find the conditional probability $P(X > 1.2 | X < \sqrt{2})$.
5. [5%] We let $Y = \text{round}(X)$. That is Y is the integer to which X is rounded. For example, if $X = \sqrt{2} \approx 1.414$, then $Y = \text{round}(X) = 1$. Another example, if $X = 1.7$, then $Y = \text{round}(X) = 2$. If $X = 2.233$, then $Y = \text{round}(X) = 2$.

   Find the probability $P(Y > X)$.

   Hint: This may be the hardest question in this exam. Come back to this question when you have time.

Question 7: [12%, Work-out question, Learning Objective 1] Consider the following experiment. Flip a fair coin, and if it is head, then stop. The entire experiment has ended.

Otherwise, if it is tail, we flip the fair coin for two more times. The entire experiment has ended. (Basically if the first flip is head, we stop. If the first flip is tail, we will end up flipping the coin three times, including the first flip.)

Consider the following two events:

$$A = \{\text{The total number of heads (in the entire experiment) is 1}\} \tag{4}$$

$$B = \{\text{The result of the very first flip is a tail}\}. \tag{5}$$

Are these two events A and B independent? Please write down very detailed arguments why A and B is or isn't independent.

Hint: If you do not know how to answer this question, please write down the definition of independence. You will receive 4 points if your answer is correct.

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$