Final Exam of ECE302, Section 3 (Prof. Wang's section)
3:30–5:30pm, Friday, May 06, 2016, STEW 130.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [12%, Work-out question] Suppose random variable X is Bernoulli distributed with parameter $p_X = 0.5$. Conditioning on $X = x$ (or equivalently after knowing the value of $X = x$), the random variable Y is Bernoulli distributed with parameter value $p_Y = 0.5 + 0.25(-1)^x$. For example, if observing $X = 0$, then the conditional distribution of Y is Bernoulli distributed with $p_Y = 0.5 + 0.25(-1)^0 = 0.75$.

1. [3%] Find the joint sample space $S_{X,Y}$.

2. [4%] Find the joint probability mass function $p_{x,y}$.

Construct another random variable $Z = (4X - 2)^Y$.

3. [5%] Let $F_Z(z)$ denote the cdf of Z. Plot $F_Z(z)$ versus z for the range of $-5 \leq z \leq 5$.

   Hint: If you do not know the answer to the previous questions, you can assume Z is binomial distributed with parameter $n = 3$ and $p = \frac{1}{3}$ and plot $F_Z(z)$ for the range of $-5 \leq z \leq 5$. You will receive 3 points if your answer is correct.

Question 2: [9%, Work-out question] Suppose we toss three *independent* fair 6-faced dice simultaneously. Let X_1, X_2, and X_3 denote the outcomes of each die, respectively. We use the outcomes X_1 to X_3 to construct the following two random variables $Y = \sum_{i=1}^{3}(-1)^i X_i$ and $W = \sum_{i=1}^{3} i \cdot X_i$. Answer the following questions.

1. [6%] Find $E(W)$ and $\mathsf{Var}(W)$.

   Hint: You can use the following fact that the outcome of a fair 6-faced die has variance $\frac{35}{12}$. There is no need to re-derive the variance of a fair die.

2. [3%] Find $\mathsf{Cov}(Y, W)$.

Question 3: [10%, Work-out question]

(X_1, X_2) is jointly Gaussian distributed with $m_1 = 2$, $m_2 = -1$, $\sigma_1^2 = 1$, and $\sigma_2^2 = 9$ and X_1 and X_2 are *orthogonal*.

1. [4%] Write down the joint pdf of X_1 and X_2.
2. [6%] Consider another random variable $Y = X_2 - 2X_1$. Find the probability of $P(|\log_2(Y)| < 2)$ in terms of the $Q(\cdot)$ function, where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$.

   Hint 1: $\log_2$ is the logarithmic function with base 2. For example, $\log_2(1) = 0$, $\log_2(2) = 1$, $\log_2(16) = 4$, and $\log_2(\frac{1}{8}) = -3$.

   Hint 2: If you do not know how to solve this question, you can find the probability $P(|X_2| < 3)$ in terms of the Q function. You will receive 4 points if your answer is correct.

Question 4: [14%, Work-out question] In average, a network server receives 0.1 webpage requests per minutes. Let X denote the (random) number of requests that the server receives between 10–11am and we assume that X is Poisson distributed. Given $X = x$, among the total number of x requests, let Y denote the (random) number of requests for video files. Obviously, $x - Y$ is the number of requests for non-video files. We assume that given $X = x$, Y is binomial distributed with $n = x$ and $p = \frac{1}{3}$.

1. [4%] Find the probability $P(X = 8, Y = 2)$.

2. [4%] Find the conditional probability $P(X = 2|Y = 1 \text{ and } X < 3)$.

   Hint: If you do not know how to solve this question, you can assume instead that X and Y are outcomes of two independent fair 6-faced dice, respectively. You will receive 3 points if your answer is correct.

3. [6%] Find the expectation $E(Y)$.

   Hint: If you do not know how to solve this question, you can assume instead that X and Y are outcomes of two independent fair 6-faced dice, respectively. But now you need to compute the following expectation $E(\pi^X Y)$ instead. You will receive 4 points if your answer is correct.

Question 5: [16%, Work-out question] A transmitter randomly chooses the X value according the following distribution

$$P(X = x) = \begin{cases} \frac{x+1}{6} & \text{if } x = 0 \text{ or } x = 1 \text{ or } x = 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Let N denote a Bernoulli distribution with $p = 0.4$. We assume X and N are independent.

A receiver will receive $Y = (X + N) \mod 3$.

(Namely, Y is the remainder when dividing $(X + N)$ by 3. We sometimes call it the *modulo* function. For example, if $X = 2$ and $N = 0$, then $Y = (2 + 0) \mod 3$ means $Y = 2$. If $X = 0$ and $N = 1$, then $Y = (0 + 1) \mod 3$ means $Y = 1$. If $X = 2$ and $N = 1$, then $Y = (2 + 1) \mod 3$ means $Y = 0$.)

1. [3%] What is the joint sample space of $S_{X,Y}$?

2. [4%] What is the joint weight assignment of (X, Y)?

   Hint: you can either use a table method or to use a tree method.

3. [3%] Given $Y = 0$, what is the ML detector $\hat{x}_{\text{ML}}$?

   Hint: If you do not know the answer to the previous subquestions, please write down what is the acronym of ML and how you plan to find an ML detector provided you know the joint pmf $p_{x,y}$. You will receive 2 points if your answer is completely correct.

4. [3%] Given $Y = 0$, what is the MAP detector $\hat{x}_{\text{ML}}$?

   Hint: If you do not know the answer to the previous subquestions, please write down what is the acronym of MAP and how you plan to find an MAP detector provided you know the joint pmf $p_{x,y}$. You will receive 2 points if your answer is completely correct.

5. [3%] Given $Y = 0$, what is the MMSE estimator $\hat{x}_{\text{MMSE}}$?

   Hint: If you do not know the answer to the previous subquestions, please write down what is the acronym of MMSE and how you plan to find an MMSE estimator provided you know the joint pmf $p_{x,y}$. You will receive 2 points if your answer is completely correct.

Question 6: [10%, Work-out question]

A continuous random variable X is uniformly distributed in the interval $(1, 3)$. Namely, X can be any real value between 1 and 3, e.g., $X = \sqrt{2}$, etc.

Y is a continuous random variable with the conditional pdf of Y given X being

$$f_{Y|X}(y|x) = \begin{cases} 1 & \text{if } 2 < x < 3 \text{ and } 1 < y < 2 \\ 1 & \text{if } 1 < x < 2 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Find the linear MMSE estimator of X given Y. Namely, find $\hat{x}_{\text{lin.MMSE}}(y)$.

Hint 1: If you do not know how to solve this question, you can find the following values instead: $E(X)$, $E(Y)$, $\mathsf{Var}(X)$, $\mathsf{Var}(Y)$, and $\rho_{X,Y}$ instead. You will receive 8 points if your answers are correct.

Hint 2: You probably want to find the joint pdf first before you answer the above questions.

Question 7: [10%, Work-out question] Consider a continuous uniform random variable X on the interval $(-1, 2)$. Also consider a random process $Y(t) = X\cos(t)$.

1. [3%] Plot three different *sample paths* of $Y(t)$ that corresponds to $X = -0.25$, $X = 0$, and $X = 1$ for the range of $-2\pi < t < 2\pi$.

2. [4%] Find the probability $P(Y(0) > 0 \text{ and } Y(\pi) > -0.5)$

3. [3%] The power of a signal over the interval $[0, 2\pi]$ is defined as

$$\mathsf{power} = \frac{1}{2\pi}\int_0^{2\pi} |Y(t)|^2 dt. \tag{3}$$

   (This definition is identical to what you have learned in ECE301.)

   Question: Find the expected power of the random process $Y(t)$ over the interval $[0, 2\pi]$. Namely, find $E(\mathsf{power})$.

   Hint 1: You may want to use the following trigonometric formula $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$.

   Hint 2: If you do not know how to solve this question, you can solve $E(Y(0) \cdot Y(\pi))$ instead. You will receive 1.5 points if your answer is correct.

Question 8: [19%, True/false question. There is no need to justify your answers]
Decide whether the following statements are true or false.

1. [2%] X and Y are uniformly distributed in a square centered at the $(2, -2)$, i.e., those (x, y) satisfying $1 < x < 3$ and $-3 < y < -1$. The random variables X and Y are negative correlated.

2. [2%] We use X_1 to X_{100} to denote 100 different random variables. Each X_i has mean zero $E(X_i) = 0$ and any two of them has covariance $\mathsf{Cov}(X_i, X_j) = 1$. Let $Y = \sum_{i=1}^{100} X_i$. We must have $E(Y) = 0$.

3. [2%] Both X and Y are Gaussian random variable and X and Y are independent. Therefore X and Y must also be orthogonal.

4. [2%] Suppose (X_1, X_2) have correlation coefficient being 0.5. Construct $Y = 100X_2$. Then the correlation coefficient between X_1 and Y is still 0.5.

5. [2%] If X and Y are independent random variables, Y and Z are independent random variables, and X and Y are independent random variables, then we always have $E(XYZ) = E(X)E(Y)E(Z)$.

6. [2%] Consider $f_Y(y)$ to be the pdf of random variable Y. We must have $0 \leq f_Y(y) \leq 1$.

7. [2%] X is a Poisson random variable with $\alpha = 10$. Let $p_x = P(X = x)$ be the corresponding pmf. We must have $\sum_{x=200}^{\infty} p_x \leq 0.05$.

8. [2%] X is a binomial random variable with $n = 100$ and $p = 0.3$. Let $p_x = P(X = x)$ be the corresponding pmf. We must have

$$\sum_{x=0}^{20} p_x + \sum_{x=40}^{\infty} p_x \leq 0.21. \tag{4}$$

9. [3%] Suppose $P(X > Y) = 0.5$ and $P(Y > Z) = 0.3$. Then $P(X > Z) = \min(0.5, 0.3) = 0.3$.

ECE 302, Summary of Random Variables

Discrete Random Variables

- Bernoulli Random Variable

  $S = \{0, 1\}$

  $p_0 = 1 - p$, $p_1 = p$, $0 \le p \le 1$.

  $E(X) = p$, $\mathrm{Var}(X) = p(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})$, $G_X(z) = (1 - p + pz)$.

- Binomial Random Variable

  $S = \{0, 1, \cdots, n\}$

  $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \cdots, n$.

  $E(X) = np$, $\mathrm{Var}(X) = np(1-p)$, $\Phi_X(\omega) = (1 - p + pe^{j\omega})^n$, $G_X(z) = (1 - p + pz)^n$.

- Geometric Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = p(1-p)^k$, $k = 0, 1, \cdots$.

  $E(X) = \frac{(1-p)}{p}$, $\mathrm{Var}(X) = \frac{1-p}{p^2}$, $\Phi_X(\omega) = \frac{p}{1-(1-p)e^{j\omega}}$, $G_X(z) = \frac{p}{1-(1-p)z}$.

- Poisson Random Variable

  $S = \{0, 1, 2, \cdots\}$

  $p_k = \frac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \cdots$.

  $E(X) = \alpha$, $\mathrm{Var}(X) = \alpha$, $\Phi_X(\omega) = e^{\alpha(e^{j\omega}-1)}$, $G_X(z) = e^{\alpha(z-1)}$.

Continuous Random Variables

- Uniform Random Variable

  $S = [a, b]$

  $f_X(x) = \frac{1}{b-a}$, $a \le x \le b$.

  $E(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $\Phi_X(\omega) = \frac{e^{j\omega b} - e^{j\omega a}}{j\omega(b-a)}$.

- Exponential Random Variable

  $S = [0, \infty)$

  $f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$ and $\lambda > 0$.

  $E(X) = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$.

- Gaussian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$.

  $E(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\Phi_X(\omega) = e^{j\mu\omega - \frac{\sigma^2\omega^2}{2}}$.

- Laplacian Random Variable

  $S = (-\infty, \infty)$

  $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$ and $\alpha > 0$.

  $E(X) = 0$, $\mathrm{Var}(X) = \frac{2}{\alpha^2}$, $\Phi_X(\omega) = \frac{\alpha^2}{\omega^2 + \alpha^2}$.

- 2-dimensional Gaussian Random Vector

  $S = \{(x, y) : \text{for all real-valued } x \text{ and } y\}$

  $$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-m_X)^2}{\sigma_X^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{\sqrt{\sigma_X^2\sigma_Y^2}} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right)}$$

  $E(X) = m_X$, $\mathrm{Var}(X) = \sigma_X^2$, $E(Y) = m_Y$, $\mathrm{Var}(Y) = \sigma_Y^2$, and $\mathrm{Cov}(X, Y) = \rho\sqrt{\sigma_X^2\sigma_Y^2}$.

- n-dimensional Gaussian Random Variable

  $S = \{(x_1, x_2, \cdots, x_n) : \text{for all real-valued } x_1 \text{ to } x_n\}$

  If we denote $\vec{x} = (x_1, x_2, \cdots, x_n)$ as an n-dimensional row-vector, then the pdf of an n-dimensional Gaussian random vector becomes

  $$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(K)}} e^{-\frac{1}{2}(\vec{x}-\vec{m})K^{-1}(\vec{x}-\vec{m})^{\mathrm{T}}}$$

  where $\vec{m}$ is the mean vector of X, i.e., $\vec{m} = E(\vec{X})$; K is an $n \times n$ covariance matrix, where the (i, j)-th entry of the K matrix is $\mathrm{Cov}(X_i, X_j)$; $\det(K)$ is the determinant of K; and K^{-1} is the inverse of K.

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$