Midterm #3 of ECE302, Prof. Wang's section
6:30–7:30pm Thursday, April 7, 2011, MTHW 210.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, e-mail address, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. There are 12 pages in the exam booklet. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

E-mail:

Signature:

Question 1: [15%, Work-out question] Consider a Gaussian random variable X that has mean $m = 3$ variance $\sigma^2 = 4$.

1. [5%] Express the probability $P(|X| > 15)$ in terms of the $Q(x)$ function. Hint: By definition $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$.
2. [5%] Use the Chebyshev inequality to upper bound the probability $P(|X-3| > 12)$.
3. [5%] What is the definition of the moment generating function $X^*(s)$?

1) $P(|X|>15) = \int_{15}^{\infty} \frac{1}{\sqrt{2\pi 4}} e^{-\frac{(x-3)^2}{2(4)}} dx + \int_{-\infty}^{-15} \frac{1}{\sqrt{2\pi 4}} e^{-\frac{(x-3)^2}{2(4)}} dx$ $\quad u = \frac{x-3}{2} \quad du = \frac{1}{2} dx$

$= \int_6^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + \int_{-\infty}^{-9} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$

$= Q(6) + (1 - Q(-9)) = Q(6) + Q(9)$

2) $P(|X - \mu_X| > k) \leq \frac{\sigma^2}{k^2}$

$P(|X-3| > 12) \leq \frac{4}{144} = \frac{1}{36}$

3) $X(s) = E[e^{-sX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(4)}} e^{-sx} e^{-\frac{(x-3)^2}{2(4)}} dx$

Question 2: [10%, Work-out question]

X is a Poisson random variable with parameter $\alpha = 3$; Y is a Gaussian random variable with $m = -2$, $\sigma^2 = \frac{9}{4}$; and X and Y are independent.

1. [10%] Find $E((X+Y)^2)$.

$$E[X^2 + 2XY + Y^2] = E[X^2] + 2E[X]E[Y] + E[Y^2]$$

$$= \left(Var(X) + (E[X])^2\right) + 2E[X]E[Y] + \left(Var(Y) + (E[Y])^2\right)$$

$$= \alpha + \alpha^2 + 2(\alpha)(-2) + \left(\frac{9}{4} + 4\right)$$

$$= 3 + 9 + (-4)(3) + \frac{25}{4}$$

$$= \frac{12 + 36 - 48 + 25}{4}$$

$$= \frac{25}{4}$$

Question 3: [20%, Work-out question]

Consider a Poisson random variable X with $\alpha = 3.5$ and given $X = x_0$, the distribution of Y is binomial with parameter $n = x_0$ and $p = \frac{1}{4}$.

1. [20%] Find the mean and variance of Y.

2. [Bonus 5%] Bonus question: What is the marginal distribution of Y?

Note: You may interpret this problem as follows: In average, every hour there are 3.5 coins being put into the vending machine. The actual number of coins may vary.

For each coin put into the vending machine, we flip that coin (with probability 1/4 being head). Y is thus the total number of heads we observe in one hour.

This interpretation could help you find the value of $E(Y)$.

1) $P(Y=k \mid X=x_0) = \binom{x_0}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{x_0-k}$

$E[Y] = E[E[Y|X]] = E[\frac{1}{4}X] = \frac{1}{4}E[X] = \frac{1}{4}\left(\frac{7}{2}\right) = \frac{7}{8}$

$E[Y^2] = E[E[Y^2|X]] = E[X(\frac{1}{4})(\frac{3}{4}) + (\frac{1}{4}X)^2] = \frac{3}{16}E[X] + \frac{1}{16}E[X^2]$

$= \frac{3}{16}\left(\frac{7}{2}\right) + \frac{1}{16}\left(\frac{7}{2} + \left(\frac{7}{2}\right)^2\right) = \frac{21}{32} + \frac{1}{16}\left(\frac{14+49}{4}\right) = \frac{42+63}{64} = \frac{105}{64}$

$Var(Y) = E[Y^2] - (E[Y])^2 = \frac{105}{64} - \frac{49}{64} = \frac{56}{64} = \frac{7}{8}$

2) Poisson, $\alpha = \frac{7}{8}$

Question 4: [25%, Work-out question] The continuous random variable Y is uniformly distributed on $(1,4)$, i.e., Y can be 1.01, 2.57, π, etc. Given $Y=y_0$, X is exponentially distributed with parameter $\lambda = y_0$.

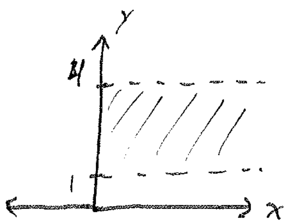
1. [20%] Find the joint cdf $F_{X,Y}(x,y)$.

   Note: If you do not know the answer of this question, you may assume that Y is uniformly distributed on $(1,4)$; X is exponentially distributed with parameter $\lambda = 5$; and X and Y are independent. If you solve $F_{X,Y}(x,y)$ using this assumption, you will still receive 16 points.

2. [5%] Express the probability $P(-3 < X < 2, 1 \le Y \le 5)$ in terms of $F_{X,Y}(x,y)$.

1) $f_Y(y) = \begin{cases} \frac{1}{3} & 1 \le y \le 4 \\ 0 & ow \end{cases}$ $\qquad f_{X|Y}(x|Y=y_0) = \begin{cases} y_0 e^{-y_0 x} & x \ge 0 \\ 0 & ow \end{cases}$

$$f_{X,Y}(x,y) = \begin{cases} \frac{y}{3} e^{-yx} & 1 \le y \le 4,\ x \ge 0 \\ 0 & ow \end{cases}$$



Case 1: $x \le 0$ or $y < 1$

$$F_{XY}(x,y) = 0$$

Case 2: $x \ge 0,\ 1 \le y \le 4$

$$F_{X,Y}(x,y) = \int_1^y \int_0^x \frac{\beta}{3} e^{-\alpha\beta}\, d\alpha\, d\beta = \int_1^y -\frac{1}{3} e^{-\alpha\beta}\Big|_0^x d\beta$$

$$= \int_1^y -\frac{1}{3}\left(e^{-\beta x} - e^{0}\right) d\beta = -\frac{1}{3}\left[-\frac{1}{x} e^{-\beta x} - \beta\right]_1^y$$

$$= \frac{1}{3x}\left(e^{-xy} - e^{-x}\right) + \frac{1}{3}(y-1)$$

Case 3: $x > 0,\ y > 4$

$$F_{X,Y}(x,y) = \int_1^4 \int_0^x \frac{\beta}{3} e^{-\alpha\beta}\, d\alpha\, d\beta = -\frac{1}{3}\left[-\frac{1}{x} e^{-\beta x} - \beta\right]_1^4$$

$$= \frac{1}{3x}\left(e^{-4x} - e^{-x}\right) + 1$$

$$F_{X,Y}(x,y) = \begin{cases} 0 & ow \\ \frac{1}{3x}\left(e^{-xy} - e^{-x}\right) + \frac{1}{3}(y-1) & x > 0,\ 1 \le y \le 4 \\ \frac{1}{3x}\left(e^{-4x} - e^{-x}\right) + 1 & x > 0,\ y > 4 \end{cases}$$

Alternate Solution:

$$f_{X,Y}(x,y) = \begin{cases} \frac{5}{3} e^{-5x} & x \geq 0,\ 1 \leq y \leq 4 \\ 0 & ow \end{cases}$$

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y) = P(X \leq x) P(Y \leq y) = F_X(x) F_Y(y)$$

$$F_X(x) = \begin{cases} 1 - e^{-5x} & x \geq 0 \\ 0 & ow \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y < 1 \\ \frac{y-1}{3} & 1 \leq y \leq 4 \\ 1 & y > 4 \end{cases}$$

$$F_{X,Y}(x,y) = \begin{cases} 1 - e^{-5x} & y > 4,\ x \geq 0 \\ \left(\frac{y-1}{3}\right)(1 - e^{-5x}) & 1 \leq y \leq 4,\ x \geq 0 \\ 0 & ow \end{cases}$$

2) $P(-3 < X < 2, 1 \leq Y \leq 5) = P(X < 2, Y \leq 5) - P(X \geq -3, Y > 1)$

$= F_{X,Y}(2^-, 5) - F_{X,Y}(-3, 5) - F_{X,Y}(2^-, 1^-) + F_{X,Y}(-3, 1^-)$

Question 5: [20%, Multiple choice question. There is no need to justify your answers]

1. [3%] X and Y are uniformly distributed in a unit disk centered at the origin, i.e., those (x, y) satisfying $x^2 + y^2 \leq 1$. Compute $R = \sqrt{X^2 + Y^2}$ as the radius and Θ is the corresponding angle (ranging from 0 to 2π). Are R and Θ independent? Yes

2. [3%] R is uniformly distributed on $(0, 1)$; Θ is uniformly distributed on $(0, 2\pi)$; and R and Θ are independent. Compute $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$. Are X and Y independent? ~~Yes~~ No

3. [2%] Consider a joint cdf $F_{X,Y}(x, y)$. Does the inequality $F_{X,Y}(1, 2) \leq F_{X,Y}(2, 3)$ always hold? Yes

4. [2%] Continue from the previous question. Does the inequality $F_{X,Y}(101, 102) < F_{X,Y}(102, 103)$ always hold? No

5. [3%] Two random variables X and Y are uncorrelated. Which one of the following statements is true: (a) The correlation between X and Y is always > 0; (b) The correlation between X and Y is always < 0; (c) The correlation between X and Y is always $= 0$; (d) none of the above. D

6. [3%] Is the following statement true? "If two random variables X and Y are positively correlated, then the expectation of the product $E(XY)$ is larger than the product of expectations $E(X)E(Y)$." Yes

7. [2%] Is the following statement true? "X is exponentially distributed with parameter $\lambda = 2$ and Z is a standard Gaussian channel. If X and Z are uncorrelated, then X and Z are orthogonal." Yes

8. [2%] Is the following statement true? "X is exponentially distributed with parameter $\lambda = 2$ and Z is a standard Gaussian channel. If X and Z are uncorrelated, then X and Z are independent." No